PRICING CALLABLE BONDS WITH STOCHASTIC INTEREST RATE AND STOCHASTIC DEFAULT RISK: A 3D FINITE DIFFERENCE MODEL

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ABSTRACT

This paper presents a 3D model for pricing defaultable bonds with embedded call options. The pricing model incorporates three essential ingredients in the pricing of defaultable bonds: stochastic interest rate, stochastic default risk, and call provision. Both the stochastic interest rate and the stochastic default risk are modeled as a square-root diffusion process. The default risk process is allowed to be correlated with the default-free term structure. The call provision is modeled as a constraint on the value of the bond in the finite difference scheme. This paper can provide new insight for future research on defaultable bond pricing models.

JEL classifications: C00; G13

Keywords: Defaultable bond; Embedded option; Partial differential equation; Finite difference method

1. INTRODUCTION

The pricing of defaultable securities has occupied a central place in the academic and practitioner literature. The standard theoretical paradigm for pricing defaultable securities is the contingent claims approach pioneered by Black and Scholes (1973). Much of the literature follows Merton (1974) by explicitly linking the risk of a firm's default to the variability in the firm's asset value. Although this line of research has proven very useful in addressing the qualitatively important aspects of pricing defaultable securities, it has been less successful in practical applications. The lack of success owes to the fact that firms' capital structures are typically quite complex and priority rules are often violated. In response to these difficulties, an alternative modeling approach has been pursued in a number of papers, including Madan and Unal (1994), Jarrow and Turnbull (1995), Duffie and Singleton (1999). At each instant, there is some probability that a firm defaults on its obligation. This is called the instantaneous probability of default. The processes of both this probability and the recovery rate determine the value of default risk. Although these processes are not formally linked to the firm's asset value, there is presumably some underlying relation, thus Duffie and Singleton describe this alternative approach as a reduced-form model (Duffee, 1999).

This paper is an effort to develop one such model in a 3D setting for pricing defaultable bonds with embedded call options. The remainder of this paper is organized as follows.

Section 2 presents the model. Section 3 describes the methodology. Section 4 concludes this paper.

2. MODEL

I derive the pricing model for defaultable bonds with embedded call options by adopting Duffie and Singleton (1999)'s reduced-form approach and Hull (2000)'s replicating-portfolio approach.

According to Duffie and Singleton, defaultable bonds can be valued by discounting at a default-adjusted interest rate, R:

$$R = r + hL\,, \tag{1}$$

where r is the risk-free interest rate, h is the hazard rate for default (i.e., the instantaneous probability of default) at time t, and L is the loss rate (i.e., the expected fractional loss in the market value) if default were to occur at time t, conditional on the information available up to time t. That is, the price at time 0 of a defaultable discount bond, f, is:

$$f = E[\exp(-\int_0^T R dt) X]\,, \tag{2}$$

where X is the face value, T is the maturity time, and E is the risk-neutral, conditional expectation at date 0. This is natural, in that hL is the risk neutral mean-loss rate of the defaultable discount bond due to default. Discounting at the default-adjusted short-term interest rate R therefore accounts for both the probability and timing of default, as well as for the effect of losses on default. A key feature of Equation (2) is that, assuming the risk neutral mean-loss rate process hL being given exogenously, standard term-structure models for default-free debt are directly applicable to defaultable debt by parameterizing R instead of r (Duffie and Singleton, 1999).

I assume that both the default-adjusted interest rate R and the hazard rate h fit a Cox, Ingersoll, and Ross (CIR)-style model (1985), a square-root diffusion model:

$$dR = a_R(b_R - R)dt + \sigma_R \sqrt{R} dz_R\,, \tag{3}$$

$$dh = a_h(b_h - h)dt + \sigma_h \sqrt{h} dz_h\,, \tag{4}$$

where dz_R and dz_h are Wiener processes, and the drift and the diffusion parameters are constants and are assumed to be known. The CIR-style model incorporates mean reversion and ensures that the default-adjusted interest rates and the hazard rates are always non-negative. As for the loss rate L, it is assumed to be a constant.

I make the assumption that there are a total of three defaultable bonds whose prices depend on the default-adjusted interest rate R and the hazard rate h. Because the three defaultable bonds are all dependent on the default-adjusted interest rate R and the hazard rate h, it follows from Ito's lemma that the price of the jth defaultable bond, f_j, follows a diffusion process:

$$df_j = \mu_j f_j dt + \sigma_{Rj} f_j dz_R + \sigma_{hj} f_j dz_h\,, \tag{5}$$

where

$$\mu_j f_j = \frac{\partial f_j}{\partial t} + \frac{\partial f_j}{\partial R} a_R (b_R - R) + \frac{\partial f_j}{\partial h} a_h (b_h - h)$$
$$+ \frac{1}{2}(\sigma_R \sqrt{R})^2 \frac{\partial^2 f_j}{\partial R^2} + \rho_{Rh} \sigma_R \sqrt{R} \sigma_h \sqrt{h} \frac{\partial^2 f_j}{\partial R \partial h} \tag{6}$$
$$+ \frac{1}{2}(\sigma_h \sqrt{h})^2 \frac{\partial^2 f_j}{\partial h^2},$$
$$\sigma_{Rj} f_j = \frac{\partial f_j}{\partial R} \sigma_R \sqrt{R}, \tag{7}$$
$$\sigma_{hj} f_j = \frac{\partial f_j}{\partial h} \sigma_h \sqrt{h}. \tag{8}$$

In these equations, μ_j is the instantaneous mean rate of return provided by f_j, σ_{Rj} and σ_{hj} are the components of the instantaneous standard deviation of the rate of return provided by f_j that may be attributed to R and h, and ρ_{Rh} is the correlation between dz_R and dz_h.

Because there are three defaultable bonds and two Wiener processes in Equation (5), it is possible to form an instantaneously riskless portfolio, $\prod$, using the defaultable bonds. Define k_j as the amount of the jth defaultable bond in the portfolio, so that

$$\prod = \sum_j k_j f_j \ . \tag{9}$$

The k_j must be chosen so that the stochastic components of the returns from the defaultable bonds are eliminated. From Equation (5) this means that

$$\sum_j k_j \sigma_{Rj} f_j = 0\,, \tag{10}$$
$$\sum_j k_j \sigma_{hj} f_j = 0\,. \tag{11}$$

The return from the portfolio is then given by

$$d\prod = \sum_j k_j \mu_j f_j dt\,. \tag{12}$$

The cost of setting up the portfolio is $\sum_j k_j f_j$. If there are no arbitrage opportunities, the portfolio must earn the default-adjusted interest rate, so that

$$\sum_j k_j \mu_j f_j = R \sum_j k_j f_j \tag{13}$$

or

$$\sum_j k_j f_j (\mu_j - R) = 0\,. \tag{14}$$

Equations (10), (11) and (14) can be regarded as three homogeneous linear equations in the k_j's. The k_j's are not all zero. From a well-known theorem in linear algebra, Equations (10), (11) and (14) can be consistent only if

$$f_j(\mu_j - R) = \lambda_R \sigma_{Rj} f_j + \lambda_h \sigma_{hj} f_j \tag{15}$$

or

$$\mu_j - R = \lambda_R \sigma_{Rj} + \lambda_h \sigma_{hj} \tag{16}$$

for λ_R and λ_h that are dependent only on the default-adjusted interest rate *R*, the hazard rate *h* and time *t*.

Substituting from Equations (6), (7) and (8) into Equation (15), I obtain

$$\begin{aligned}&\frac{\partial f_j}{\partial t}+\frac{\partial f_j}{\partial R}a_R(b_R-R)+\frac{\partial f_j}{\partial h}a_h(b_h-h)+\frac{1}{2}(\sigma_R\sqrt{R})^2\frac{\partial^2 f_j}{\partial R^2}\\&+\rho_{Rh}\sigma_R\sqrt{R}\sigma_h\sqrt{h}\frac{\partial^2 f_j}{\partial R\partial h}+\frac{1}{2}(\sigma_h\sqrt{h})^2\frac{\partial^2 f_j}{\partial h^2}-Rf_j\\&=\lambda_R\frac{\partial f_j}{\partial R}\sigma_R\sqrt{R}+\lambda_h\frac{\partial f_j}{\partial h}\sigma_h\sqrt{h}\end{aligned} \tag{17}$$

that reduces to

$$\begin{aligned}&\frac{\partial f_j}{\partial t}+\frac{\partial f_j}{\partial R}[a_R(b_R-R)-\lambda_R\sigma_R\sqrt{R}]+\frac{\partial f_j}{\partial h}[a_h(b_h-h)-\lambda_h\sigma_h\sqrt{h}]\\&+\frac{1}{2}(\sigma_R\sqrt{R})^2\frac{\partial^2 f_j}{\partial R^2}+\rho_{Rh}\sigma_R\sqrt{R}\sigma_h\sqrt{h}\frac{\partial^2 f_j}{\partial R\partial h}+\frac{1}{2}(\sigma_h\sqrt{h})^2\frac{\partial^2 f_j}{\partial h^2}\\&-Rf_j=0.\end{aligned} \tag{18}$$

Dropping the subscripts to *f*, I deduce that any defaultable bond whose price, *f*, is contingent on the default-adjusted interest rate, *R*, the hazard rate, *h*, and time, *t*, satisfies the second-order differential equation

$$\begin{aligned}&\frac{\partial f}{\partial t}+\frac{\partial f}{\partial R}[a_R(b_R-R)-\lambda_R\sigma_R\sqrt{R}]+\frac{\partial f}{\partial h}[a_h(b_h-h)-\lambda_h\sigma_h\sqrt{h}]\\&+\frac{1}{2}(\sigma_R\sqrt{R})^2\frac{\partial^2 f}{\partial R^2}+\rho_{Rh}\sigma_R\sqrt{R}\sigma_h\sqrt{h}\frac{\partial^2 f}{\partial R\partial h}+\frac{1}{2}(\sigma_h\sqrt{h})^2\frac{\partial^2 f}{\partial h^2}\qquad\text{Q.E.D.}\\&-Rf=0.\end{aligned} \tag{19}$$

On a coupon date, the bond value must jump by the amount of the coupon payment. Hence, to incorporate coupon payments into the model, I impose a jump condition:

$$f(R,h,t_C^-)=f(R,h,t_C^+)+K_C, \tag{20}$$

where a coupon of K_C is received at time t_C.

Bonds often have a call feature which gives the issuing company the right to purchase back the bond at any time during specified periods for a specified amount. According to the no-arbitrage argument, to incorporate a call feature into the model, I must impose a constraint on the bond's value:

$$f(R,h,t_D)\le X_D \tag{21}$$

where X_D is the call price and t_D is the call date.

To find a unique solution of Equation (19), I must impose one final condition and four boundary conditions.

The final condition corresponds to the payoff at maturity and so for a coupon-paying bond:

$$f(R,h,T) = P_T + K_T\,, \tag{22}$$

where a principal amount of P_T and a coupon payment of K_T are received at maturity.

The first boundary condition, when the default-adjusted interest rate, *R*, approaches to zero percent, can be stated as:

$$\begin{aligned} f(R,h,t) &= f(R,h,T)e^{-R(T-t)} \\ &= f(R,h,T)\,. \end{aligned} \tag{23}$$

The second boundary condition, when the default-adjusted interest rate, *R*, approaches to infinity, can be stated as:

$$\begin{aligned} f(R,h,t) &= f(R,h,T)e^{-R(T-t)} \\ &= 0. \end{aligned} \tag{24}$$

The third boundary condition, when the hazard rate, *h*, approaches to zero percent, can be stated as:

$$\begin{aligned} f(R,h,t) &= f(R,h,T)e^{-R(T-t)} \\ &= f(R,h,T)e^{-(r+hL)(T-t)} \\ &= f(R,h,T)e^{-r(T-t)}. \end{aligned} \tag{25}$$

The forth boundary condition, when the hazard rate, *h*, approaches to infinity, can be stated as:

$$\begin{aligned} f(R,h,t) &= f(R,h,T)e^{-R(T-t)} \\ &= f(R,h,T)e^{-(r+hL)(T-t)} \\ &= 0. \end{aligned} \tag{26}$$

3. METHODOLOGY

I solve the pricing model for defaultable bonds with embedded call options by a 3D explicit finite difference method (Hull, 2003; Wilmott, 2000).

Suppose that the number of months to maturity is *T*. I divide this into *L* equally spaced intervals of length $\Delta t = T / L$. Δt is fixed at one month. A total of $L+1$ times are, therefore, considered:

$$0,\ \Delta t,\ 2\Delta t,\ \ldots,\ T.$$

Suppose that h_{max} is a hazard rate sufficiently high that, when it is reached, the bond has virtually no value. I define $\Delta h = h_{max} / M$ and consider a total of $M+1$ equally spaced hazard rates:

$$0,\ \Delta h,\ 2\Delta h,\ \ldots,\ h_{max.}$$

Δh is set to be one percent.

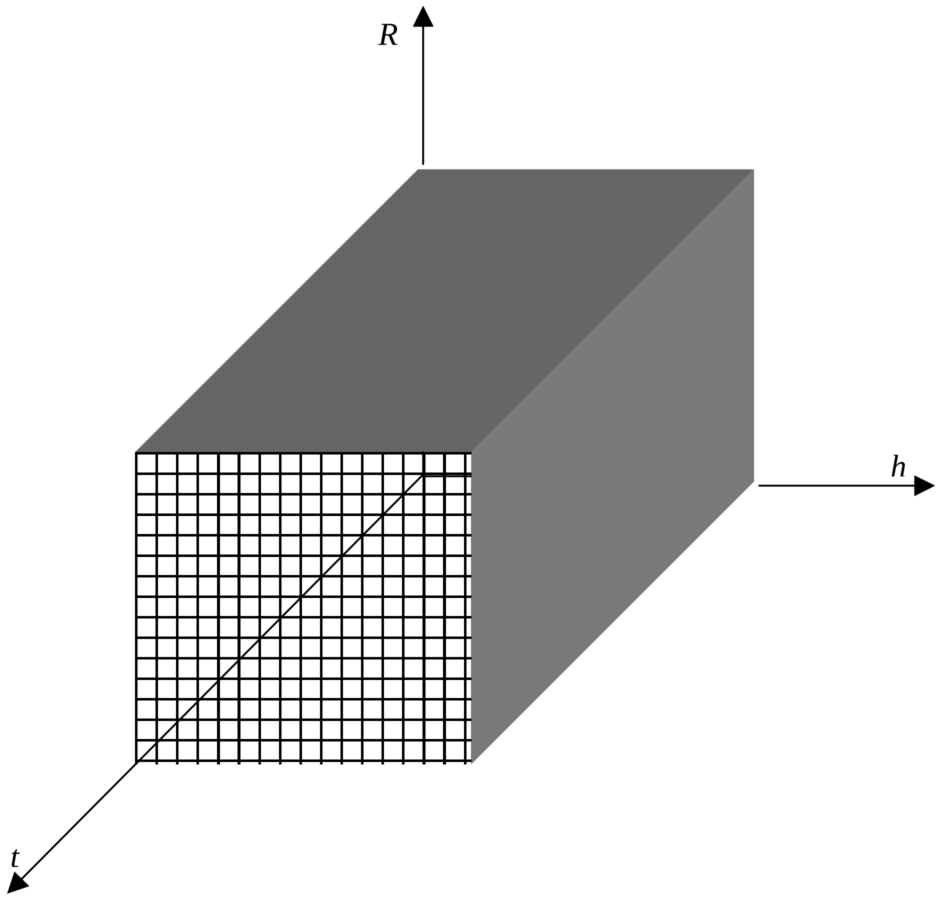
Suppose that R_{max} is a default-adjusted interest rate sufficiently high that, when it is reached, the bond has virtually no value. I define $\Delta R = R_{max} / N$ and consider a total of $N+1$ equally spaced default-adjusted interest rates:

$$0,\ \Delta R, 2\Delta R, \ldots, R_{max.}$$

ΔR is set to be one percent.

The time points, hazard rate points and default-adjusted interest rate points define a 3D grid consisting of a total of *(L+1)(M+1)(N+1)* points as shown in Figure 1. The *(i, j, k)* point on the 3D grid is the point that corresponds to default-adjusted interest rate $i\Delta R$, hazard rate $j\Delta h$ and time $k\Delta t$. I use the variable $f_{i,j}^{k}$ to denote the value of the bond at the *(i, j, k)* point.

FIGURE 1
The 3D finite difference grid



Recall that the differential equation for the price of a defaultable bond, *f(R, h, t)*, is given as:

$$\begin{aligned}&\frac{\partial f}{\partial t}+\frac{\partial f}{\partial R}[a_R(b_R-R)-\lambda_R\sigma_R\sqrt{R}]+\frac{\partial f}{\partial h}[a_h(b_h-h)-\lambda_h\sigma_h\sqrt{h}]\\&+\frac{1}{2}(\sigma_R\sqrt{R})^2\frac{\partial^2 f}{\partial R^2}+\rho_{Rh}\sigma_R\sqrt{R}\sigma_h\sqrt{h}\frac{\partial^2 f}{\partial R\partial h}+\frac{1}{2}(\sigma_h\sqrt{h})^2\frac{\partial^2 f}{\partial h^2}\\&-Rf=0.\end{aligned} \tag{27}$$

For an interior point *(i, j, k)* in the 3D grid, $\frac{\partial f}{\partial t}$ can be approximated by using a symmetric central difference:

$$\frac{\partial f}{\partial t} = \frac{f_{i,j}^{k} - f_{i,j}^{k+1}}{\Delta t}, \tag{28}$$

$\frac{\partial f}{\partial R}$ can be approximated by using a symmetric central difference:

$$\frac{\partial f}{\partial R} = \frac{f_{i+1,j}^{k} - f_{i-1,j}^{k}}{2\Delta R}, \tag{29}$$

$\frac{\partial f}{\partial h}$ can be approximated by using a symmetric central difference:

$$\frac{\partial f}{\partial h} = \frac{f_{i,j+1}^{k} - f_{i,j-1}^{k}}{2\Delta h}, \tag{30}$$

$\frac{\partial^2 f}{\partial R^2}$ can be approximated by using a symmetric central difference:

$$\frac{\partial^2 f}{\partial R^2} = \frac{f_{i+1,j}^{k} - 2 f_{i,j}^{k} + f_{i-1,j}^{k}}{\Delta R^2}, \tag{31}$$

$\frac{\partial^2 f}{\partial R \partial h}$ can be approximated by using a symmetric central difference:

$$\frac{\partial^2 f}{\partial R \partial h} = \frac{f_{i+1,j+1}^{k} - f_{i+1,j-1}^{k} - f_{i-1,j+1}^{k} + f_{i-1,j-1}^{k}}{4\Delta R \Delta h}, \tag{32}$$

and $\frac{\partial^2 f}{\partial h^2}$ can be approximated by using a symmetric central difference:

$$\frac{\partial^2 f}{\partial h^2} = \frac{f_{i,j+1}^{k} - 2 f_{i,j}^{k} + f_{i,j-1}^{k}}{\Delta h^2}. \tag{33}$$

Substituting equations (28), (29), (30), (31), (32) and (33) into the differential equation (27) and noting that $R = i\,\Delta R$, $h = j\,\Delta h$ and $f = f_{i,j}^{k}$, the corresponding difference equation can be shown as:

$$\begin{aligned}
&\frac{f_{i,j}^{k} - f_{i,j}^{k+1}}{\Delta t} + \frac{f_{i+1,j}^{k} - f_{i-1,j}^{k}}{2\Delta R}[a_R(b_R - i\Delta R) - \lambda_R \sigma_R \sqrt{i\Delta R}] \\
&+ \frac{f_{i,j+1}^{k} - f_{i,j-1}^{k}}{2\Delta h}[a_h(b_h - j\Delta h) - \lambda_h \sigma_h \sqrt{j\Delta h}] \\
&+ \frac{1}{2}(\sigma_R \sqrt{i\Delta R})^2 \frac{f_{i+1,j}^{k} - 2 f_{i,j}^{k} + f_{i-1,j}^{k}}{\Delta R^2} \\
&+ \rho_{Rh} \sigma_R \sqrt{i\Delta R} \sigma_h \sqrt{j\Delta h} \frac{f_{i+1,j+1}^{k} - f_{i+1,j-1}^{k} - f_{i-1,j+1}^{k} + f_{i-1,j-1}^{k}}{4\Delta R \Delta h} \\
&+ \frac{1}{2}(\sigma_h \sqrt{j\Delta h})^2 \frac{f_{i,j+1}^{k} - 2 f_{i,j}^{k} + f_{i,j-1}^{k}}{\Delta h^2} - (i\Delta R) f_{i,j}^{k} = 0,
\end{aligned} \tag{34}$$

where $i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, M$ and $k = 0, 1, \ldots, L$. Rearranging terms, this equation becomes:

$$\begin{aligned} &A_{i,j} f_{i,j}^{k} + B_i (f_{i+1,j}^{k} + f_{i-1,j}^{k}) + C_j (f_{i,j+1}^{k} + f_{i,j-1}^{k}) \\ &+ D_{i,j} (f_{i+1,j+1}^{k} + f_{i+1,j-1}^{k} + f_{i-1,j+1}^{k} + f_{i-1,j-1}^{k}) = f_{i,j}^{k+1}, \end{aligned} \tag{35}$$

where

$$A_{i,j} = 1 - \frac{1}{\Delta R^2}(\sigma_R \sqrt{i\Delta R})^2 \Delta t - \frac{1}{\Delta h^2}(\sigma_h \sqrt{j\Delta h})^2 \Delta t - (i\Delta R)\Delta t\,,$$

$$B_i = \frac{1}{2\Delta R}[a_R(b_R - i\Delta R) - \lambda_R \sigma_R \sqrt{i\Delta R}]\Delta t + \frac{1}{2\Delta R^2}(\sigma_R \sqrt{i\Delta R})^2 \Delta t\,,$$

$$C_j = \frac{1}{2\Delta h}[a_h(b_h - j\Delta h) - \lambda_h \sigma_h \sqrt{j\Delta h}]\Delta t + \frac{1}{2\Delta h^2}(\sigma_h \sqrt{j\Delta h})^2 \Delta t\,,$$

$$D_{i,j} = \frac{1}{4\Delta R \Delta h} \rho_{Rh} \sigma_R \sqrt{i\Delta R} \sigma_h \sqrt{j\Delta h} \Delta t\,,$$

$i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, M$ and $k = 0, 1, \ldots, L$.

The value of the bond at time T is $P_T + K_T$, where P_T is the principal amount and K_T is the coupon payment. Hence,

$$f_{i,j}^{k} = P_T + K_T \tag{36}$$

for $i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, M-1$ and $k = 0$.

The value of the bond when the default-adjusted interest rate is zero percent is $f(R,h,T)$. Hence,

$$f_{i,j}^{k+1} = f_{i,j}^{k} \tag{37}$$

for $i = 0$, $j = 0, 1, \ldots, M-1$ and $k = 0, 1, \ldots, L-1$.

I assume that the bond is worth zero when the default-adjusted interest rate is one hundred percent, so that

$$f_{i,j}^{k+1} = 0 \tag{38}$$

for $i = N$, $j = 0, 1, \ldots, M-1$ and $k = 0, 1, \ldots, L-1$.

The value of the bond when the hazard rate is zero percent is $f(R,h,T)e^{-r(T-t)}$. Hence,

$$f_{i,j}^{k+1} = f_{i,j}^{k} e^{-r(T-t)} \tag{39}$$

for $i = 1, 2, \ldots, N-1$, $j = 0$ and $k = 0, 1, \ldots, L-1$.

I assume that the bond is worth zero when the hazard rate is one hundred percent, so that

$$f_{i,j}^{k+1} = 0 \tag{40}$$

for $i = 0, 1, \ldots, N$, $j = M$ and $k = -1, 0, \ldots, L-1$.

To incorporate coupon payments into the model, I impose a jump condition. Hence,

$$f_{i,j}^{k} = f_{i,j}^{k} + K_C \tag{41}$$

for $i = 0, 1, \ldots, N\text{-}1$, $j = 0, 1, \ldots, M\text{-}1$, $k = t_C$ or the coupon date and K_C is the coupon payment.

To incorporate call features into the model, I impose a constraint on the bond's value. Hence,

$$f_{i,j}^{k} \le X_D \tag{42}$$

for $i = 0, 1, \ldots, N\text{-}1$, $j = 0, 1, \ldots, M\text{-}1$, $k = t_D$ or the call date and X_D is the call price.

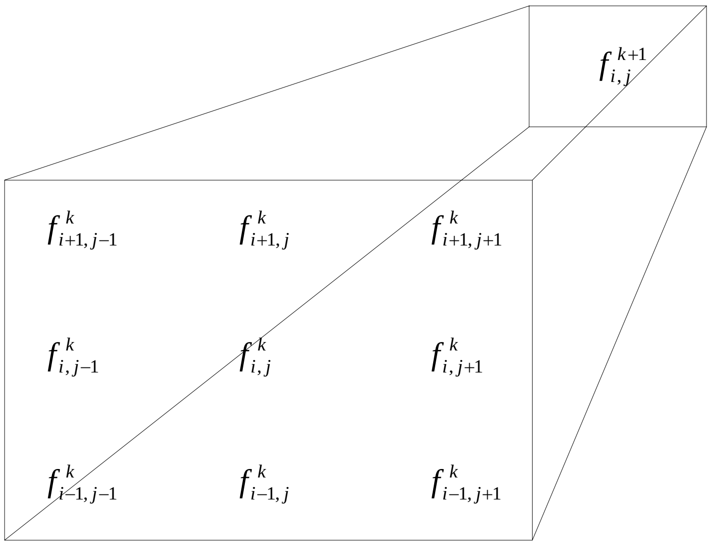
Equations (36), (37), (38), (39) and (40) define the value of the bond along the five planes of the 3D grid in Figure 1, where $t = T$, $R = 0\%$, $R = 100\%$, $h = 0\%$ and $h = 100\%$. Equation (35) defines the value of the bond at all other points.

Equation (35) shows that there are nine known bond values linked to one unknown bond value. See Figure 2. Hence, for each time layer there are $(N\text{-}1)(M\text{-}1)$ equations in $(N\text{-}1)(M\text{-}1)$ unknowns; the boundary conditions yield the values at the four boundaries for each time layer and the final condition gives the values in the last time layer.

To find the bond value of interest, go backwards in time, solving for a sequence of linear equations. Eventually, $f_{1,1}^{L}$, $f_{1,2}^{L}$, $f_{1,3}^{L}$, …, $f_{N-1,M-1}^{L}$ are obtained. One of these is the bond price of interest. If the initial default-adjusted interest rate or the initial hazard rate does not lie on the grid point, I use a linear interpolation between the two bond prices on the neighboring grid points to find the bond price of interest.

FIGURE 2

The relationship between bond values in the 3D explicit finite difference method



4. CONCLUSION

This paper presents a 3D model for pricing defaultable bonds with embedded call options. The pricing model incorporates three essential ingredients in the pricing of defaultable bonds: stochastic interest rate, stochastic default risk, and call provision. Both the stochastic interest rate and the stochastic default risk are modeled as a square-root diffusion process. The default risk process is allowed to be correlated with the default-free term structure. The call provision is modeled as a constraint on the value of the bond in the finite difference scheme. The model is by no means a complete success. To improve the model, one can assume that the recovery rate in the event of default varies stochastically through time. In summary, this paper can provide new insight for future research on defaultable bond pricing models.

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