The instantaneous and forward default intensity of structural models

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ABSTRACT:

When the value of a firm is observable, Duffie and Lando (2001) and Giesecke (2005) show that the default intensity does not exist unless the time of default is totally inaccessible. When the firm value follows a diffusion process or a jump-diffusion process, the probability that an announcing sequence approaches the default time strictly from below is not zero. Therefore, the default intensity does not exist. They conclude that the structural approach does not lay theoretical groundwork for hazard-rate based estimation of default intensities. In this paper, we refer to this default intensity as the instantaneous default intensity, which is the expected rate of default occurrence in the immediately next moment given the information up to current time.

When the firm value follows a diffusion process with a fixed default threshold, it is well known that the time of default follows an inverse Gaussian distribution. For a first passage model, the probability density function of the default time divided by the survival function, represents the expected rate of default occurrence. We show that the conditional default probability is a function of this intensity. However, for a diffusion process, there should not be an associated intensity that leads to the conditional default probability. We call this paradox the intensity paradox.

The answer to the intensity paradox is the existence of the forward intensity, which is defined as the expected rate of default occurrence in the future given the information up to current time. We show that the forward intensity does possibly exist for the accessible stopping time, the totally inaccessible stopping, and the stopping time neither accessible nor totally inaccessible, if some other conditions are satisfied. For a jump-diffusion process, we show that the forward default intensity can be separated as the sum of the forward default intensity caused by diffusion and the forward default intensity caused by a jump.

For a structural model, we point out that it should be the forward default intensity, not the instantaneous default intensity, that is used to calculated the conditional probability of default. We show how to calculate the conditional default probability in terms of the forward intensity. The yield spread and the forward spread are both obtained in terms of forward intensities. We conclude that the structural model does define an intensity process, a forward one but not always the instantaneous one.

Key words and phrases: Credit risk; Default risk; Diffusion process; Jump-diffusion process; Structural model; Intensity process; Hazard rate.

1 Introduction

When the value of a firm is observable, Duffie and Lando (2001) and Giesecke (2005) show that the default intensity does not exist unless the time of default is totally inaccessible. When the firm value follows a diffusion process or a jump-diffusion process, the probability that an announcing sequence approaches the default time strictly from below is not zero. Therefore, the default intensity does not exist. They conclude that the structural approach does not lay theoretical groundwork for hazard-rate based estimation of default intensities. In this paper, we refer to this default intensity as the instantaneous default intensity, which is the expected rate of default occurrence in the immediately next moment given the information up to current time. We further explain these results in section 3 and section 4.

When the firm value follows a diffusion process with a fixed default threshold, it is well known that the time of default follows an inverse Gaussian distribution. For a first passage model, the probability density function of the default time divided by the survival function, represents the expected rate of default occurrence. We show that the conditional default probability is a function of this intensity. However, for a diffusion process, there should not be an associated intensity that leads to the conditional default probability. We call this paradox the intensity paradox and show it in section 5.

The answer to the intensity paradox is the existence of the forward intensity, which is defined as the expected rate of default occurrence in the future given the information up to current time. The definition of the forward intensity is given in section 6. The existence of the default intensity is examined in section 7. We show that the forward intensity does possibly exist for the accessible stopping time, the totally inaccessible stopping, and the stopping time neither accessible nor totally inaccessible, if some other conditions are satisfied. For a jump-diffusion process, we show that the forward default intensity can be separated as the sum of the forward default intensity caused by diffusion and the forward default intensity caused by a jump.

For a structural model, we point out that it should be the forward default intensity, not the instantaneous default intensity, that is used to calculated the conditional probability of default. We show how to calculate the conditional default probability in terms of the forward intensity. In section 8, we obtain the yield spread and the forward in terms of forward intensities. We conclude that the structural model does define an intensity process, a forward one but not always the instantaneous one.

2 Model setup

Let $\Omega = \{\omega\}$ be the space of the elementary events and $\mathcal{F}$ be a collection of subsets $A \subseteq \Omega$ called events, forming a sigma-algebra. Let Q be the probability measure on Ω. All random variables are defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q)$, which is a probability space $(\Omega, \mathcal{F}, Q)$ endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $0 \leq s < t$ and $\mathcal{F}_t = \mathcal{F}_{t+}$. We assume that $\mathcal{F}_0$ contains all Q-null sets. At time $t > 0$, $\mathcal{F}_t$ can be regarded as the sigma-algebra of events observable on the time interval $[0, t]$.

When a firm is not able to fulfill its financial obligations, we call it default. Let the time of default is denoted by τ. τ is also referred to as the first passage default time. Assume that the cumulative distribution function and probability density function exist and are denoted by $F_\tau(t)$ and $f_\tau(t)$ respectively. If

$$\{\tau \leq t\} \in \mathcal{F}_t, \ \forall \ t \geq 0,$$

τ is called a Markov time or a stopping time, which means that we know whether default has occurred or not at every time t.

We assume that investors are able to observe the value of a public firm from past to current time. This assumption is referred to as the complete information by Duffie and Lando (2001) and Giesecke (2005). In practice, the value of a public firm can possibly be obtained by adding up the market price of the equity and the liability if these data are available. We treat the firm value as if it were a traded instrument. Let $\{V_s\}_{s\geq 0}$ be a stochastic process where V_s denotes the firm value at time s. We assume that the firm value is right continuous with left limits. We assume the the firm value is continuous in probability with independent and stationary increments. At current time t, investors can observe firm values $V_s = v_s, 0 \leq s \leq t$ while $\{V_s\}_{s>t}$ remains unknown because no one can foresee the future trajectory of the firm value.

The σ-algebra on $\Omega \times \mathbb{R}_+$ generated by all sets of the form

$$A \times \{0\}, \ A \in \mathcal{F}_0$$

and

$$A \times (s, t], \ 0 \leq s < t < \infty, \ A \in \mathcal{F}_s,$$

is called the predictable $\sigma-$algebra for the filtration $(\mathcal{F})_{t\geq 0}$. A process X is called predictable with respect to a filtration if, as a mapping from $\Omega \times \mathbb{R}_+$ to $\mathbb{R}$, it is measurable with respect to the predictable $\sigma-$algebra generated by that filtration. A stopping time is predictable if $\{(\omega, t) : t \in \mathbb{R}_+, 0 \leq t < \tau(\omega)\}$ is a predictable set. A predictable firm-value process can be regarded as a process whose behavior at time t is determined by information on $[0, t)$. When the firm value follows a diffusion process, it is a predictable process. When the firm value follows a jump-diffusion process or a pure jump process, it is not a predictable process.

For the structural model, we define the time of default in a standard way as the first time the firm value hit its default threshold [Black and Cox (1976), Leland (1994), Longstaff and Schwartz (1995), or Zhou (1997)]. It is defined by $\tau = \inf\{t > 0 : V_t \leq D\}$ where D is a known default threshold with a trivial condition $V_0 > D$. When default occurs, the firm is liquidated at or after the time of default to pay off its financial obligations. The solvency ratio process is defined by $\left\{U(t) = \ln\left(\frac{V_t}{D}\right)\right\}_{t \geq 0}$. The default time can be also defined by $\tau = \inf\{t > 0, U(t) \leq 0\}$. Let $L(t_1, t_2] = \inf\{U(t), t_1 < t \leq t_2, t_1 < t_2\}$ be the minimum solvency ratio during the time interval $(t_1, t_2]$. The corresponding model filtration is $\mathcal{F}_t = \sigma\{V_s, 0 \leq s \leq t\}, t \geq 0$. Because

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t,$$

τ is a stopping time. If a stopping time is predictable, then there exists a nondecreasing sequence of stopping times $\tau_1 \leq \tau_2 \leq \cdots$ with $\tau_n < \tau$ and $\lim_{n \to \infty} \tau_n = \tau$ for $\forall\ \omega \in \Omega$ with $\{\tau(\omega) > 0\}$. This sequence is called an announcing sequence for τ. The graph of a stopping time is defined by $[\tau] = \{(\omega, t), t \in \mathbb{R}_+, t = \tau(\omega)\}$. A stopping is called accessible if there is a sequence $\{\tau_n\}_{n \geq 1}$ of predictable times such that $[\tau] \subseteq \cup_n [\tau_n]$. A stopping time is called totally inaccessible if $Q\{\tau = \tau' < \infty\} = 0$ for each predictable stopping time τ'. Therefore, no predictable stopping time can give any information about the totally inaccessible stopping time. Some stopping times are neither accessible nor totally inaccessible.

When default will never occur, the default time is denoted by $\tau = \infty$. For each stopping time $\tau < \infty$, there exits one and only one (to within Q-negligibility) pair of an accessible stopping time τ_a and a totally inaccessible stopping time τ_i such that $[\tau] = [\tau_a] \cup [\tau_i]$ and $[\tau_a] \cap [\tau_i] = \emptyset$. (see Liptser and Shiryaev, 1998, theorem 3.3). One can say that the classes of accessible stopping times and totally inaccessible stopping times are "orthogonal".

Consider a structural default model. Each default time $\tau < \infty$ can be decomposed into τ_a and τ_i. Because $[\tau_a]$ and $[\tau_i]$ are disjoint, the default model can be regarded as a double-decrement context (see Bowers, N.L. eds., 1997, Ch 10). If $\tau < \infty$, then the default time follows $\tau = \min\{\tau_a, \tau_i\} = \tau_a \wedge \tau_i$. If default is caused by a predictable process, then $\tau_a < \infty$ and $\tau_i = \infty$. Otherwise, $\tau_i < \infty$ and $\tau_a = \infty$.

Accessibility and the decomposition property play important roles in determine whether the default intensity exists. This issue is explained in the next section.

3 The instantaneous default intensities of the first passage default time

In the previous section, we assume that the cumulative distribution function and probability density function exist and are denoted by $F_\tau(t)$ and $f_\tau(t)$ respectively. The hazard rate function of τ is defined by

$$h(T) = \frac{f_\tau(T)}{1 - F_\tau(T)},\ T \geq 0.$$

With information about the firm value up to time $t > 0$, $\mathcal{F}_t$, the conditional hazard rate function is given by

$$h(T|t) = \frac{f_\tau(T|t)}{1 - F_\tau(T|t)} = \frac{f_\tau(T|t)}{S_\tau(T|t)},\ T \geq t > 0,$$

where $F_\tau(T|t) = Q\{\tau \leq T|\mathcal{F}_t\}$, $S_\tau(T|t) = 1 - F_\tau(T|t)$, and $f_\tau(T|t) = \frac{\partial F_\tau(T|t)}{\partial T}$. The conditional hazard rate function equals

$$h(T|t) = \frac{f_\tau(T|t)}{Q\{\tau > T|\mathcal{F}_t\}} = \lim_{h \downarrow 0} \frac{Q[T < \tau \leq T + h \mid \tau > T, \mathcal{F}_t]}{h},\ T \geq t > 0,$$

The conditional probability density function of the default time thus equals

$$f_\tau(T|t) = Q\{\tau > T|\mathcal{F}_t\} h(T|t) = e^{-\int_t^T h(s|t)ds} h(T|t).$$

We consider the first passage model and do not treat default as recurrent events. The number of default is defined by

$$N(t) = 1_{\{\tau \leq t\}}.$$

This one jump process is a step function which is right continuous with left limits. Default can occur only if default has not occurred in the past. The expected rate of default occurrence given the firm has not defaulted with information $\mathcal{F}_t$, is given by

$$\mu(t \mid \mathcal{F}_t) = \lim_{h \downarrow 0} \frac{\mathrm{E}[N(t+h) - N(t) \mid \tau > t, \mathcal{F}_t]}{h}. \tag{1}$$

We call it the instantaneous default intensity conditioned on survival. Because $N(t)$ can take values 0 or 1, so

$$\begin{aligned}
\mu(t \mid \mathcal{F}_t) &= \lim_{h \downarrow 0} \frac{1 \times Q[N(t+h) - N(t) = 1 \mid \tau > t, \mathcal{F}_t] + 0 \times Q[N(t+h) - N(t) = 0] \mid \tau > t, \mathcal{F}_t]}{h} \\
&= \lim_{h \downarrow 0} \frac{Q[N(t+h) - N(t) = 1 \mid \tau > t, \mathcal{F}_t]}{h} \\
&= \lim_{h \downarrow 0} \frac{Q[t < \tau \leq t + h \mid \tau > t, \mathcal{F}_t]}{h}.
\end{aligned}$$

When the firm value is observable, $\mu(t \mid \mathcal{F}_t)$ is a time-dependent function. When the firm value is not observable, $\mu(t \mid \mathcal{F}_t)$ is a stochastic process depending on the realization of the event ω. In this paper, we are working on the case when the firm value is observable. Therefore, $\mu(t \mid \mathcal{F}_t)$ is a time-dependent function. The relationship between the instantaneous default intensity conditioned on survival and the conditional hazard rate is given by

$$\mu(t \mid \mathcal{F}_t) = h(t|t),\ t \geq 0.$$

$\{N(t)\}_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ with $\mathrm{E}[N(t)] < \infty$, for $\forall\ t < \infty$. Because $\mathrm{E}[N(t+s)|\mathcal{F}_t] \geq N(t)$ a.s. for $\forall\ s \geq 0,\ t \geq 0$, $\{N(t)\}_{t\geq 0}$ is a right-continuous nonnegative submartingale. The Doob-Meyer decomposition theorem states that there exists a unique (to within indistinguishability) nondecreasing right-continuous predictable process $\{A(t)\}_{t\geq 0}$ such that

$$M(t) = N(t) - A(t),$$

$\{M(t)\}_{t\geq 0}$ is a martingale. $\{A(t)\}_{t\geq 0}$ is called the predictable compensator of $\{N(t)\}_{t\geq 0}$. If $A(t)$ is finite and differentiable ($\lim_{h\downarrow 0} \frac{A(t+h)-A(t)}{h}$ exists and is finite) with $A'(t) = \lambda(t)$, then $\{\lambda(t)\}_{t\geq 0}$ is called the intensity process for $\{N(t)\}_{t\geq 0}$. The

$$A(t) = \int_0^t \lambda(s)ds,$$

is called the cumulative intensity process for $\{N(t)\}_{t\geq 0}$. The $dA(t) = \lim_{h\downarrow 0}[A(t+h) - A(t)]$ can be regarded as the average rate of growth of $N(t)$ with information up to time t.

If $\mu(t \mid \mathcal{F}_t)$ exists and $\mu(t \mid \mathcal{F}_t) < \infty$ for $0 \leq t < \infty$, we consider

$$M(t) = N(t) - \int_0^t 1_{\{\tau>s\}}\mu(s \mid \mathcal{F}_s)ds,\ t \geq 0.$$

For $\forall\ t \geq 0$,

$$\begin{aligned}
\mathrm{E}[M(t)] &= \mathrm{E}\left[1_{\{\tau\leq t\}}\right] - \mathrm{E}\left[\int_0^t 1_{\{\tau>s\}}\mu(s \mid \mathcal{F}_s)ds\right] \\
&= Q\{\tau \leq t\} - Q\{\tau \leq t\} \\
&= 0.
\end{aligned}$$

We can see that $M(t)$ is a martingale and $\int_0^t 1_{\{\tau>s\}}\mu(s \mid \mathcal{F}_s)ds$ is $\mathcal{F}_t$-predictable. If $\int_0^t 1_{\{\tau>s\}}\mu(s|s)ds$ exists, by the uniqueness (to within indistinguishability) of the predictable compensator, we have

$$\lambda(t) = 1_{\{\tau>t\}}\mu(t \mid \mathcal{F}_t) \ \text{ a.s..} \tag{2}$$

The intensity can be also defined by

$$\lambda(t) = \lim_{h\downarrow 0} \frac{\mathrm{E}[N(t+h) - N(t) \mid \mathcal{F}_t]}{h} = \lim_{h\downarrow 0} \frac{Q\{t < \tau < t+h \mid \mathcal{F}_t\}}{h},$$

with

$$\mathrm{E}\left[\int_0^t \lambda(s)ds\right] = Q\{\tau \leq t\}.$$

This intensity is the rate of default arrival in the immediately next moment given information up to time t. Therefore, we call it instantaneous default intensity in this paper.

When the $N(t)$ is not a one jump process and τ can recur, then $1_{\{\tau>t\}}$ will drop out of equation (2). For example, when $N(t)$ is a homogeneous Poisson process with parameter λ, then $\lim_{h\downarrow 0}\frac{Q\{t<\tau<t+h|\mathcal{F}_t\}}{h}=\lambda$.

$\lambda(t)$ represents the expected rate of the growth of $N(t)$ given information up to time t, while $\mu(t\mid\mathcal{F}_t)$ in equation (1), represents the expected rate of growth of $N(t)$ given default has not occurred with information up to time t. The existence of the intensity is examined in the next section.

4 The existence of instantaneous default intensity

4.1 Accessible stopping times

When the stopping time is accessible, $N(t)=1_{\{\tau\leq t\}}$ is predictable. When $N(t)$ is predictable, $A(t)=N(t)$ (see Liptser and Shiryaev, 1998, theorem 3.12). The derivative of $A(t)$ is $A'(t)=0$ for $t<\tau$ and $A'(t)=\infty$ for $t=\tau-$. This violates the necessary conditions for the existence the default intensity that $A(t)$ is differentiable for $t\geq 0$ and is finite. Therefore, the instantaneous default intensity process $\{\lambda(t)\}_{t\geq 0}$ does not exist.

Given $\mathcal{F}_t$ at time t, investors know $V_s=v_s$ for $0\leq s\leq t$. For $V_s,\ s>t$, investors know its probability density function but not what will exactly occur. Because the firm value evolution has independent and stationary increments, the instantaneous default intensity conditioned on survial equals

$$\begin{aligned}\mu(t\mid\mathcal{F}_t) &= \lim_{h\downarrow 0}\frac{Q\{\tau<t+h\mid\tau>t,\mathcal{F}_t\}}{h}\\ &= \lim_{h\downarrow 0}\frac{Q[V_{t+h}\leq D\mid L[0,t]>0,V_t=v_t]}{h}.\end{aligned}$$

Because the stopping is accessible, there exists a announcing sequence, such as $\tau_n=\inf\{t>0,V_t\leq D+1/n\}$, giving early warning signals of default with $\lim_{n\to\infty}\tau_n=\tau$. By looking at the latest τ_n, investors can be certain of whether default will occur or not in the immediate next moment. If default will not occur, then $\mu(t\mid\mathcal{F}_t)=0$. If default will occur, then $\mu(t\mid\mathcal{F}_t)=\infty$. Because of the accessibility, given that the firm has not defaulted at time t, the range of the firm value at time t can be classified into $V^D=\{v_t:\lim_{h\downarrow 0}Q\{V_{t+h}\leq D|V_t=v_t\}=1,\ t\geq 0\}$ and $\overline{V^D}=\{v_t:\lim_{h\downarrow 0}Q\{V_{t+h}\leq D|V_t=v_t\}=0,\ t\geq 0\}$.

$$\mu(t\mid\mathcal{F}_t) = \begin{cases} 0, & v_t\in\overline{V^D}\\ \infty, & v_t\in V^D.\end{cases}$$

For example, consider the case when the firm value follows a diffusion process. If $v_t > D$, the firm value will not to diffuse D in the immediate next moment. If $v_t = D+$, the firm value will diffuse to D with probability one. Therefore,

$$\mu(t \mid \mathcal{F}_t) = \begin{cases} 0, & v_t > D \\ \infty, & v_t = D+ . \end{cases}$$

The instantaneous default intensity conditioned on survival does not exist when the stopping time is accessible.

4.2 Totally inaccessible stopping times

Duffie and Lando (2001) state that "for a stopping time to have an associated intensity, it must (among other properties) be totally inaccessible, meaning that, for any sequence of stopping time, the probability that the sequence approaches τ strictly from below is zero." Based on $\mathcal{F}_t$, no matter what firm value V_t is, investors cannot assert that default will occur ($\mu(t \mid \mathcal{F}_t) = \infty$) nor can they assert that default will not occur ($\mu(t \mid \mathcal{F}_t) = 0$) in the immediately next moment. It must be the case $0 < \mu(t \mid \mathcal{F}_t) < \infty$. If this is violated ($\mu(t \mid \mathcal{F}_t) = 0$ or ∞) for some specific firm values, then there exist a sequence of stopping times such that the probability that the sequence approaches τ strictly from below is larger than zero. Mathematically speaking, a totally inaccessible stopping must satisfies

$$\lim_{h \downarrow 0} Q\{\tau < t + h \mid \tau > t, \mathcal{F}_t\} = O(h), \forall\, t \geq 0.$$

The instantaneous default intensity conditioned on survival equals the probability density function that $-dV_t$ is larger or equals to the distance to default $V_t - D$ as

$$\begin{aligned} \mu(t \mid \mathcal{F}_t) &= \lim_{h \downarrow 0} \frac{Q\{\tau < t + h \mid \tau > t, \mathcal{F}_t\}}{h} \\ &= \lim_{h \downarrow 0} \frac{Q[V_{t+h} \leq D \mid L[0,t] > 0, V_t = v_t]}{h}, \\ &= \lim_{h \downarrow 0} \frac{Q[V_{t+h} - V_t \leq D - V_t \mid L[0,t] > 0, V_t = v_t]}{h}. \end{aligned}$$

Because we assume the firm value evolution has independent and stationary increments,

$$\begin{aligned} \mu(t \mid \mathcal{F}_t) &= \lim_{h \downarrow 0} \frac{Q[V_h \leq D - v_t \mid V_0 = 0]}{h} \\ &= \lim_{h \downarrow 0} \int_{-\infty}^{D - v_t} f_{V_h}(x) dx, \end{aligned}$$

where $\int_{-\infty}^{\infty} f_{V_h}(x) dx = 1$ [1]. For the one jump process, we can conclude that

$$0 < \mu(t \mid \mathcal{F}_t) < 1, \forall\, t \geq 0,$$

[1] For an accessible stopping, such as a diffusion type firm-value model, $\lim_{h \downarrow 0} f_{V_h}(x) = 0, x \leq D - v_t$ if $v_t > D$. $\lim_{h \downarrow 0} f_{V_h}(x) = \begin{cases} 0, & x < D - v_t \\ \infty, & x = D - v_t, \end{cases}$ if $v_t = D+$. Therefore, the instantaneous default intensity does not exist.

meaning that the expected rate of default arrival is less than one. The instantaneous default intensity equals

$$\lambda(t) \quad = \quad 1_{\{\tau>t\}} \int_{-\infty}^{D-v_t} f_{V_h}(x)dx.$$

The compensator for $N(t)$,

$$A(t) = \int_0^t 1_{\{\tau>t\}} \mu(s \mid \mathcal{F}_s) ds < t < \infty, \forall\ 0 \le t < \infty.$$

For example, the stopping time is totally inaccessible when the firm value follows a compound Poisson process as $V_t = \sum_{i=1}^{N_t} X_i$, where N_t is a Poisson counting process representing the number of jumps up to time t with parameter λ and X_i is the jump size of the firm value for $i-th$ jump and is independent of N_t with CDF $F_X(x)$ and PDF $f_X(x)$ for $-\infty < x < \infty$. We have

$$Q\{\tau < t+h \mid \tau > t, \mathcal{F}_t\} = \lambda h F_X(D - v_t) + o(h) = O(h), \forall\ t \ge 0.$$

The instantaneous default intensity conditioned on survival thus equals

$$\begin{aligned} \mu(t \mid \mathcal{F}_t) &= \lim_{h \downarrow 0} \frac{\lambda h F_X(D - v_t)}{h} \\ &= \lambda F_X(D - v_t). \end{aligned}$$

4.3 Stopping times neither accessible nor totally inaccessible

There are stopping times which are neither accessible nor totally inaccessible, meaning that, there exists at least a sequence of stopping times approaching τ strictly from below with probability larger than 0 and smaller than 1. Because everything stopping time can be decomposed into an accessible stopping time and a totally inaccessible stopping time, the one jump process $N(t)$ follows

$$N(t) = 1_{\{(\tau_a \wedge \tau_i) \le t\}} = 1_{\{\tau_a \le t\}} + 1_{\{\tau_i \le t\}}.$$

$1_{\{\tau_i \le t\}}$ has an associated instantaneous intensity process, however, $1_{\{\tau_a \le t\}}$ does not. Therefore, we cannot find an associated intensity process for a stopping time which is neither totally inaccessible nor accessible. When the firm value follows a jump-diffusion process, such as Zhou (1997), the stopping time is neither accessible nor totally inaccessible. When the firm value is way from the default threshold, default can occur only caused by a jump. Investors cannot assert whether default will occur or not. The instantaneous intensity is finite. When the firm value is about to cross the default threshold at $V_t = D+$, the firm will diffuse to default threshold with probability 1. The intensity is ∞. Therefore, we cannot find an associated intensity process for a stopping time which is neither totally inaccessible nor accessible.

5 Intensity paradox

In this section, we consider the case when the firm value follows a diffusion process as

$$dV_t = rV_t dt + \sigma V_t dZ_t, \tag{3}$$

where the rate of return r equals the risk-less rate, σ^2 is the variance, Z_t is a standard Brownian motion under the Q measure with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. The default time $\tau = \inf\{t > 0, V_t \leq D\}$ is an accessible stopping time. Therefore, there does not exist an associated instantaneous intensity process $\{\lambda(t)\}_{t\geq 0}$ so that $Q\{\tau \leq t\} = \mathrm{E}\left[\int_0^t \lambda(s)ds\right]$.

Consider at current time t, the investors know the firm value $\{V_s = v_s\}_{\{s\leq t\}}$. Investors do not know the firm values V_s for $s > t$ but know the distribution of the firm value evolution in the future. If the firm has not defaulted in the past, the distribution function of τ given $\mathcal{F}_t$ can be obtained using the result of Harrison (1990) as

$$\begin{aligned} F_\tau(T \mid \mathcal{F}_t) &= \Phi\left(-\frac{\ln\left(\frac{v_t}{D}\right) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad + \exp\left(-\frac{2\ln\left(\frac{v_t}{D}\right)(r - \frac{\sigma^2}{2})}{\sigma^2}\right)\Phi\left(-\frac{\ln\left(\frac{v_t}{D}\right) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\ &= \Phi\left(-\frac{\ln\left(\frac{v_t}{D}\right) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad + \left(\frac{v_t}{D}\right)^{1-\frac{2r}{\sigma^2}}\Phi\left(-\frac{\ln\left(\frac{v_t}{D}\right) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right), \quad t < T < \infty. \end{aligned}$$

The probability mass function of the event that default will never occur, is given by

$$Q\{\tau = \infty \mid \mathcal{F}_t\} = 1 - \exp\left(-\frac{2\ln\left(\frac{v_t}{D}\right)(r - \frac{\sigma^2}{2})}{\sigma^2}\right) = 1 - \left(\frac{v_t}{D}\right)^{1-\frac{2r}{\sigma^2}}.$$

The probability density function of τ is the derivative of $F_\tau(T \mid \mathcal{F}_t)$ as

$$f_\tau(T \mid \mathcal{F}_t) = \frac{\ln\left(\frac{v_t}{D}\right)}{\sqrt{2\pi\sigma^2}(T-t)^{\frac{3}{2}}}\exp\left\{-\frac{\left[\ln\left(\frac{v_t}{D}\right) + (r - \frac{\sigma^2}{2})(T-t)\right]^2}{2\sigma^2(T-t)}\right\}, \quad \tau < \infty.$$

Conditioned on $\tau < \infty$, τ follows an inverse Gaussian distribution. [2] The hazard rate function can be obtained by

$$h(T \mid \mathcal{F}_t) = \frac{f_\tau(T \mid \mathcal{F}_t)}{1 - F_\tau(T \mid \mathcal{F}_t)}$$

[2] Conditioned on $\tau < \infty$, the probability density function of τ is

$$f_\tau(T \mid \tau < \infty, \mathcal{F}_t) = \frac{f_\tau(T \mid \mathcal{F}_t)}{Q(\tau < \infty \mid \mathcal{F}_t)} = \frac{\ln\left(\frac{v_t}{D}\right)}{\sqrt{2\pi\sigma^2}(T-t)^{\frac{3}{2}}}\exp\left\{-\frac{\left[\ln\left(\frac{v_t}{D}\right) - (r - \frac{\sigma^2}{2})(T-t)\right]^2}{2\sigma^2(T-t)}\right\},$$

which is an inverse Gaussian probability density function.

$$= \frac{\frac{\ln\left(\frac{v_t}{D}\right)}{\sqrt{2\pi\sigma^2}(T-t)^{\frac{3}{2}}} \exp\left\{-\frac{\left[\ln\left(\frac{v_t}{D}\right)+(r-\frac{\sigma^2}{2})(T-t)\right]^2}{2\sigma^2(T-t)}\right\}}{\Phi\left(\frac{\ln\left(\frac{v_t}{D}\right)-(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) - \left(\frac{v_t}{D}\right)^{1-\frac{2r}{\sigma^2}} \Phi\left(-\frac{\ln\left(\frac{v_t}{D}\right)-(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)}, \tag{4}$$

which represents the expected default occurrence at time s given the firm survives time s with information up to time t, where $0 \le t < s$. The probability of default given default has not occurred is

$$Q\{t < \tau \le T \mid \tau > t, \mathcal{F}_t\} = \int_t^T e^{-\int_t^s h(z|\mathcal{F}_t)dz} h(s \mid \mathcal{F}_t)ds. \tag{5}$$

We can define a process $\{\lambda(s|t)\}_{s\ge t}$, which is similar to the default intensity process, by

$$\lambda(s|t) = 1_{\{\tau > s|\mathcal{F}_t\}} h(s \mid \mathcal{F}_t), \ s > t.$$

$$\begin{aligned} \mathrm{E}\left[\int_t^T \lambda(s|t)ds\right] &= \int_t^T Q\{\tau > s|\mathcal{F}_t\} h(s \mid \mathcal{F}_t)ds \\ &= Q\{t < \tau \le T \mid \tau > t, \mathcal{F}_t\}. \end{aligned}$$

Because the time of default is an accessible stopping time, there does not exist an associated instantaneous default intensity process $\{\lambda(t)\}_{t\ge 0}$ so that $\mathrm{E}\left[\int_0^T \lambda(s)ds\right] = Q\{\tau \le T\}, \ T > 0$. Why does there exist a process $\{\lambda(s|0)\}_{s>0}$ so that $\mathrm{E}\left[\int_0^T \lambda(s|0)ds\right] = Q\{\tau \le T \mid \mathcal{F}_0\}$?

6 The forward intensity of the first passage model

At time t, the conditional probability of default occurring in $(t,T]$, $Q\{t < \tau \le T \mid \tau > t, \mathcal{F}_t\}$ in equation (5), is assessed based on $\mathcal{F}_t$ for the fixed t, but not based on the progressively filtration $(\mathcal{F}_s)_{t<s\le T}$. The instantaneous default intensity conditional on survival $\mu(t|\mathcal{F}_t)$ and the instantaneous default intensity $\lambda(t)$ are both assessed based on $\mathcal{F}_t$ progressively in time. It should not be the $\mu(t|\mathcal{F}_t)$ nor $\lambda(t)$ that is used to calculate $Q\{t < \tau \le T \mid \tau > t, \mathcal{F}_t\}$ because the information available is only up to time t. It should be a forward intensity based on $\mathcal{F}_t$ for fixed t, that is used to calculated $Q\{t < \tau \le T \mid \tau > t, \mathcal{F}_t\}$.

The forward expected rate of default occurrence at time $T > t$, given default has not occurred by time T, is defined by

$$\mu(T \mid \mathcal{F}_t) = \lim_{h\downarrow 0} \frac{\mathrm{E}[N(T+h) - N(T) \mid \tau > T, \mathcal{F}_t]}{h}, \ 0 \le t < T < \infty.$$

We call it the forward default intensity conditioned on future survival. For the one jump process, this intensity equals

$$\mu(T \mid \mathcal{F}_t) \;=\; \lim_{h\downarrow 0} \frac{1 \times Q\{N(T+h) - N(T) = 1 \mid \tau > T, \mathcal{F}_t\} + 0 \times Q\{N(T+h) - N(T) = 0 \mid \tau > T, \mathcal{F}_t\}}{h}$$

$$
\begin{aligned}
&= \lim_{h\downarrow 0} \frac{Q\{N(T+h)-N(T)=1 \mid \tau > T, \mathcal{F}_t\}}{h} \\
&= \lim_{h\downarrow 0} \frac{Q\{T < \tau \leq T+h \mid \tau > T, \mathcal{F}_t\}}{h} \\
&= h(T|t), \ 0 \leq t < T < \infty.
\end{aligned}
$$

Therefore, the forward default intensity conditioned on future survival equals the conditional hazard rate. If this forward default intensity exists, then we consider the following process, a process assessed based on $\mathcal{F}_t$ for fixed t,

$$
M(T|t) = N(T|t) - A(T|t) = 1_{\{t<\tau\leq T|\tau>t,\mathcal{F}_t\}} - \int_t^T 1_{\{\tau>s|\tau>t,\mathcal{F}_t\}}\mu(s|\mathcal{F}_t)ds, \tag{6}
$$

for $0 \leq t < T < \infty$. This process is assessed based on a fixed filtration $\mathcal{F}_t$ for a fixed t, given that the firm has not default by time t. This is not the Doob-Meyer decomposition, which is a decomposition progressively based on $(\mathcal{F}_t)_{t\geq 0}$. Taking the expected value of equation (6), we have

$$
\begin{aligned}
\mathrm{E}[M(T|t)] &= \mathrm{E}\left[1_{\{t<\tau\leq T|\tau>t,\mathcal{F}_t\}}\right] - \mathrm{E}\left[\int_t^T 1_{\{\tau>s|\tau>t,\mathcal{F}_t\}}\mu(s|\mathcal{F}_t)ds\right] \\
&= Q\{t<\tau\leq T|\tau>t,\mathcal{F}_t\} - Q\{t<\tau\leq T|\tau>t,\mathcal{F}_t\} \\
&= 0,
\end{aligned}
$$

which is a $\mathcal{F}_t$-martingale for fixed t. If $A'(T|t)$ exists and is finite, we call it the forward default intensity conditioned on current survival. If the intensity exists, then

$$
dA(T|t) = \mathrm{E}[N(T+dt)-N(T)|\tau>t,\mathcal{F}_t],
$$

and

$$
\begin{aligned}
\lambda(T|t) &= A'(T|t) \\
&= 1_{\{\tau>T|\tau>t,\mathcal{F}_t\}}\mu(T|\mathcal{F}_t) \\
&= \lim_{h\downarrow 0} \frac{\mathrm{E}[N(T+h)-N(T) \mid \tau>t,\mathcal{F}_t]}{h}, \ 0 \leq t < T < \infty.
\end{aligned}
$$

The existence of the forward intensity is examined in the next section.

7 The existence of the forward intensity

7.1 The accessible stopping time

When a stopping time is accessible, investors can assert whether default will occur at time $t+dt$ given $\mathcal{F}_t, \ t \geq 0$. Given that the firm has not defaulted at time t, the set of all firm value at time t can be separated into $V^D = \{v_t : \lim_{h\downarrow 0} Q\{V_{t+h} \leq D|V_t = v_t\} = 1, \ t \geq 0\}$ and $\overline{V^D} = \{v_t : \lim_{h\downarrow 0} Q\{V_{t+h} \leq D|V_t = v_t\} = 0, t \geq 0\}$. For a structural model,

$$
\mu(T \mid \mathcal{F}_t) = \lim_{h\downarrow 0} \frac{Q\{L(t,T]>0, V_{T+h} \leq D \mid V_t = v_t, L[0,T]>0\}}{h}, \ 0 \leq t < T < \infty.
$$

Because the firm value evolution has independent and stationary increments,

$$\begin{aligned}\mu(T \mid \mathcal{F}_t) &= \lim_{h\downarrow 0} \frac{Q\{V_s > D, t < s \le T, V_{T-t+h} \le D \mid V_0 = v_t, Vs > D, s \le T-t\}}{h} \\ &= \lim_{h\downarrow 0} \frac{Q\{V_s > D - v_t, t < s \le T, V_{T-t+h} \le D - v_t \mid V_0 = 0, Vs > D, s \le T-t\}}{h},\end{aligned}$$

for $0 \le t < T < \infty$. Due to the accessibility of the stopping, investors can assert whether default will occur or not at time $T + dT$ given the firm value V_T. Let $V^D(0, T-t] = \{\omega : V_s > D - v_t, t < s \le T, \lim_{h\downarrow 0} Q\{V_{T-t+h} \le D - v_t\} = 1\}$ and $\overline{V^D}(0, T-t] = \{\omega : V_s > D - v_t, t < s \le T, \lim_{h\downarrow 0} Q\{V_{T-t+h} \le D - v_t\} = 0\}$. The forward intensity can be written by

$$\begin{aligned}\mu(T \mid \mathcal{F}_t) &= \frac{\int_{V^D(0,T-t]} f_{V_{T-t}}(x)dx}{Q\{V^D \cup \overline{V^D}\}} \\ &= \frac{f_{\tau(T|t)}}{S_\tau(T|t)} < \infty,\ 0 \le t < T < \infty.\end{aligned}$$

The forward intensity conditioned on future survival exists. For a diffusion process, it is the conditional probability density function of the firm value at time T equal $D+$, given that default has not occurred. We can conclude that

$$Q\{T < \tau \le T + h \mid \tau > T, \mathcal{F}_t\} = O(h),\ \text{as } h \downarrow 0.$$

7.2 Totally inaccessible stopping time

When a stopping time τ is totally inaccessible, for any stopping time τ', $Q\{\tau = \tau'\} = 0$. For a structural model, investors cannot assert whether the default event $\lim_{h\downarrow 0} 1_{\{t<\tau\le t+h|V_t=v_t\}}$ will occur or not. For $T > t$, neither can investors predictive whether the event $\lim_{h\downarrow 0} 1_{\{T<\tau\le T+h|\tau>T,V_t=v_t\}}$ will occur or not. Conditioned on $\{\omega : \tau > T, V_t = v_t\}$, no matter what value v_t is, the $\mu(T \mid \mathcal{F}_t) = \lim_{h\downarrow 0} \frac{Q\{T<\tau\le T+h|\tau>T,\mathcal{F}_t\}}{h}$ should follow $0 < \mu(T \mid \mathcal{F}_t) < \infty$.[3] Because $\mu(s|\mathcal{F}_t)$, for $s > t$, is finite, we can let $A(T|t) = \int_t^T 1_{\{\tau>s|\tau>t,\mathcal{F}_t\}}\mu(s|\mathcal{F}_t)ds$. Then, $A(T|t)$ is differentiable and

$$A'(T|t) = 1_{\{\tau>T|\tau>t,\mathcal{F}_t\}}\mu(T|\mathcal{F}_t),$$

is finite. Therefore, the forward intensity exists.

7.3 Stopping times neither accessible nor totally inaccessible

When the stopping time is neither accessible nor totally inaccessible, the stopping τ can be decomposed as $\tau = \tau_a \wedge \tau_i$. Default can occur caused by either an accessible process or an totally

[3] $\mu(T \mid \mathcal{F}_t) = 0$ means that default will not occur with certainty. $\mu(T \mid \mathcal{F}_t) = \infty$ means that default will occur with some certainty. Both are not properties of a totally inaccessible stopping time.

inaccessible process. The one jump process $N(T|t) = 1_{\{t<\tau\leq T|\tau>t,\mathcal{F}_t\}}$ can be decomposed as

$$\begin{aligned} N(T|t) &= 1_{\{t<\tau\leq T|\tau>t,\mathcal{F}_t\}} && (7)\\ &= 1_{\{t<\tau_a\leq T|\tau>t,\mathcal{F}_t\}} + 1_{\{t<\tau_i\leq T|\tau>t,\mathcal{F}_t\}}. && (8)\\ &= N_a(T|t) + N_i(T|t), \end{aligned}$$

where $N_a(T|t) = 1_{\{t<\tau_a\leq T|\tau>t,\mathcal{F}_t\}}$ and $N_i(T|t) = 1_{\{t<\tau_i\leq T|\tau>t,\mathcal{F}_t\}}$. Default events can be classified into two disjoint groups, default events caused by an accessible process and default events caused by an totally inaccessible process. We can consider the default model as a double-decrement model. Let J be the cause of default (see Bowers, N.L. eds., 1997, Ch 10) where

$$J = \begin{cases} a, & \tau = \tau_a \\ i, & \tau = \tau_i \end{cases} .$$

Taking expectation of equation (7) and (8), we have

$$\begin{aligned} Q\{t<\tau\leq T|\tau>t,\mathcal{F}_t\} &= Q\{t<\tau_a\leq T|\tau>t,\mathcal{F}_t\} + Q\{t<\tau_i\leq T|\tau>t,\mathcal{F}_t\}\\ &= Q\{t<\tau\leq T, J=a|\tau>t,\mathcal{F}_t\} + Q\{t<\tau\leq T, J=i|\tau>t,\mathcal{F}_t\}. \end{aligned}$$

By total probability, the forward default intensity conditioned on future survival can be written by

$$\begin{aligned} \mu(T\mid\mathcal{F}_t) &= \lim_{h\downarrow 0}\frac{Q\{T<\tau\leq T+h\mid\tau>T,\mathcal{F}_t\}}{h}\\ &= \lim_{h\downarrow 0}\left[\frac{Q\{T<\tau\leq T+h, J=a\mid\tau>T,\mathcal{F}_t\}}{h} + \frac{Q\{T<\tau\leq T+h, J=i\mid\tau>T,\mathcal{F}_t\}}{h}\right]\\ &= \lim_{h\downarrow 0}\frac{\mathrm{E}[N_a(T+h)-N_a(T)\mid\tau>T,\mathcal{F}_t]}{h} + \lim_{h\downarrow 0}\frac{\mathrm{E}[N_i(T+h)-N_i(T)\mid\tau>T,\mathcal{F}_t]}{h}\\ &= \mu^{(a)}(T\mid\mathcal{F}_t) + \mu^{(i)}(T\mid\mathcal{F}_t),\ 0\leq t<T<\infty, \end{aligned}$$

where the $\mu^{(a)}(T\mid\mathcal{F}_t) = \lim_{h\downarrow 0}\frac{\mathrm{E}[N_a(T+h)-N_a(T)|\tau>T,\mathcal{F}_t]}{h}$ represents the forward intensity of default caused by an accessible process, conditioned on future survival, while the $\mu^{(i)}(T\mid\mathcal{F}_t) = \lim_{h\downarrow 0}$ $\frac{\mathrm{E}[N_i(T+h)-N_i(T)|\tau>T,\mathcal{F}_t]}{h}$ represents the forward intensity of default caused by a totally inaccessible process, conditioned on future survival.

The forward default intensity conditioned on current survival is

$$\lambda(T|t) = 1_{\{\tau>T|\tau>t,\mathcal{F}_t\}}\mu(T|\mathcal{F}_t) = 1_{\{\tau>T|\tau>t,\mathcal{F}_t\}}\left[\mu^{(a)}(T|\mathcal{F}_t) + \mu^{(i)}(T|\mathcal{F}_t)\right]$$

The probability of default given current firm value and that default has not occurred at time t, is

$$\begin{aligned} Q\{t<\tau\leq T|\tau>t,\mathcal{F}_t\} &= \mathrm{E}\left[\int_t^T \lambda(s|t)ds\right]\\ &= \int_t^T e^{-\int_t^s\left[\mu^{(a)}(z|\mathcal{F}_t)+\mu^{(i)}(z|\mathcal{F}_t)\right]dz}\left[\mu^{(a)}(s|\mathcal{F}_t)+\mu^{(i)}(s|\mathcal{F}_t)\right]ds. \end{aligned}$$

We have to point out that the default probability, conditioned on information up to current time, is a function of the forward intensity, not the instantaneous intensity. Some authors state that the structural does not lay theoretical groundwork for hazard-rate based estimation of default intensities. This statement is not correct because it is not the instantaneous default intensity, that we should use to calculate the conditional default probability. It is the forward intensity that should be used to calculate the conditional default probability.

8 The credit spread and the forward intensity

When default occurs, the ratio of debt recovered is usually referred to as the recovery rate. We denote it by $\delta(\tau)$. A risky bond of one-dollar face amount can be regarded as a contingent claim paying one dollar if default does not occur and paying the recovery rate if default occurs. We assume that there exists a risk neutral probability measure. Under this risk-neutral probability measure, the risky bond's price at time t is the time-t expected value of the contingent claim. Let $V(t,T)$ represent the time-t price of a defaultable corporate zero-coupon bond paying one dollar at maturity time T for $0 \le t \le T < \infty$, with a yield to maturity of $R(t,T)$. The risky bond price can be written as $V(t,T) = e^{-R(t,T)(T-t)}$. Let $P(t,T)$ represent the time-t price of a default-free zero-coupon bond paying one dollar at maturity time T, with a yield to maturity $r(t,T)$. We have $P(t,T) = e^{-r(t,T)(T-t)}$. A time-$t$ value of a bank account process or the value of one dollar accumulated from time 0 to time t is defined by $B(t) = e^{\int_0^t r(s)ds}$ where $r(s)$ is the short rate at time s. When the short rate is assumed to be flat, we have $B(t) = e^{rt}$.

We assume that the recovery rate is ratio of the market value of the firm at default to its debt obligation and is payable to the debt holders at maturity. This recovery scheme is usually referred to as the recovery-of-treasury-value (RTV) scheme. The time-t price of the risky bond of one-dollar face amount can be written by

$$V(t,T) \quad = \quad \mathrm{E}_t\left[\frac{B(t)}{B(T)}\left[\delta(\tau)1_{(\tau\le T|\tau>t,\mathcal{F}_t)} + 1_{(\tau>T|\tau>t,\mathcal{F}_t)}\right]\right],$$

where $\mathrm{E}_t(\cdot)$ denotes the conditional expectation $\mathrm{E}_t(\cdot|\mathcal{F}_t)$ under the probability measure Q. If the riskless rate is independent of the default process, it can be shown that

$$V(t,T) \quad = \quad P(t,T) \times \left[1 - \{1 - \mathrm{E}_t[\delta(\tau)]\}\, Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}\right]. \tag{9}$$

Taking log value of equation (9) and dividing it by $-(T-t)$, the yield spread follows

$$R(t,T) - r(t,T) \quad = \quad -\frac{1}{T-t}\ln\left[1 - \{1 - \mathrm{E}_t[\delta(\tau) \mid t < \tau \le T]\}Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}\right]. \tag{10}$$

To link equation (10) with the default intensity, the yield spread equals

$$R(t,T) - r(t,T)$$

$$
\begin{aligned}
&= -\frac{1}{T-t} \ln \left(1 - \{1 - \mathrm{E}_t[\delta(\tau) \mid t < \tau \le T]\} \mathrm{E} \left\{ \int_t^T \lambda(s|t) ds \right\}\right) && (11) \\
&= -\frac{1}{T-t} \ln \left(1 - \{1 - \mathrm{E}_t[\delta(\tau) \mid t < \tau \le T]\} \mathrm{E}[A(T|t)]\right) \\
&= -\frac{1}{T-t} \ln \left(1 - \{1 - \mathrm{E}_t[\delta(\tau) \mid t < \tau \le T]\} \int_t^T e^{-\int_t^s \mu(z|\mathcal{F}_t) dz} \mu(s|\mathcal{F}_t) ds\right). && (12)
\end{aligned}
$$

8.1 When the firm value follows a diffusion process

When the firm value follows a diffusion process, default occurs once the firm value hits the default threshold D. The ratio $\frac{V_\tau}{D}$ equals one so a partial recovery rate is impossible. In order to link the structural model with the credit spread, we have to assume that there exist liquidation and transaction costs specifically for the case when the firm value follows a diffusion process. Therefore, the recovery rate $\delta(\tau)$ is an exogenous random variable. We further assume that the recovery rate is not a function of the maturity T. Assume that the firm value evolution is governed by equation (3) as $dV_t = rV_t dt + \sigma V_t dZ_t$. The yield spread follows equation (11) and equation (12) with $\lambda(s|t) = 1_{\{\tau > T | \tau > t, \mathcal{F}_t\}} \mu(T|\mathcal{F}_t)$ and $\mu(T|\mathcal{F}_t) = \frac{\frac{\ln\left(\frac{v_t}{D}\right)}{\sqrt{2\pi\sigma^2}(T-t)^{\frac{3}{2}}} \exp\left\{-\frac{\left[\ln\left(\frac{v_t}{D}\right) + (r - \frac{\sigma^2}{2})(T-t)\right]^2}{2\sigma^2(T-t)}\right\}}{\Phi\left(\frac{\ln\left(\frac{v_t}{D}\right) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) - \left(\frac{v_t}{D}\right)^{1-\frac{2r}{\sigma^2}} \Phi\left(-\frac{\ln\left(\frac{v_t}{D}\right) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)}$ from equation (4).

Let the forward rate for a riskless zero-coupon bond be defined by $f(t,T) \equiv -\frac{\partial}{\partial T} \ln P(t,T)$ and the forward rate for a risky zero-coupon bond is defined by $F(t,T) \equiv -\frac{\partial}{\partial T} \ln V(t,T)$. The forward spread for a reduced-form model is obtained by Chen (2005). For a structural model, the forward spread is shown in the following theorem:

Theorem 8.1 *If the default-free spot rates and the default process are independent and $E_t[\delta(\tau)]$ is not a function of T in a RTV scheme, then the forward spread is*

$$
F(t,T) - f(t,T) = \mu(T|\mathcal{F}_t) \left\{1 - E_t[\delta(\tau)] \frac{P(t,T)}{V(t,T)}\right\}. \tag{13}
$$

Proof:
From equation (9), we have

$$
\begin{aligned}
F(t,T) &= -\frac{\partial}{\partial T} \ln V(t,T) \\
&= -\frac{\partial}{\partial T} \ln \{P(t,T) \times [1 - \{1 - \mathrm{E}_t[\delta(\tau)]\} Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}]\} \\
&= f(t,T) - \frac{\partial}{\partial T} \ln [1 - \{1 - \mathrm{E}_t[\delta(\tau)]\} Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}] \\
&= f(t,T) - \frac{\frac{\partial}{\partial T} [1 - \{1 - \mathrm{E}_t[\delta(\tau)]\} Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}]}{1 - \{1 - \mathrm{E}_t[\delta(\tau)]\} Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}} \\
&= f(t,T) + \frac{\frac{\partial}{\partial T} [\{1 - \mathrm{E}_t[\delta(\tau)]\} Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}]}{1 - \{1 - \mathrm{E}_t[\delta(\tau)]\} Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}}
\end{aligned}
$$

$$
\begin{aligned}
&= f(t,T) + \frac{\{1-\mathrm{E}_t[\delta(\tau)]\}\,\frac{\partial}{\partial T}\,[Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}]}{1-\{1-\mathrm{E}_t[\delta(\tau)]\}\,Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}} \\
&= f(t,T) + \frac{\{1-\mathrm{E}_t[\delta(\tau)]\}\,e^{-\int_t^T \mu(z|\mathcal{F}_t)dz}\mu(T|\mathcal{F}_t)}{1-\{1-\mathrm{E}_t[\delta(\tau)]\}\,Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}} \\
&= f(t,T) + \mu(T|\mathcal{F}_t)\frac{\{1-\mathrm{E}_t[\delta(\tau)]\}\,[1-Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}]}{1-\{1-\mathrm{E}_t[\delta(\tau)]\}\,Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}} \\
&= f(t,T) + \mu(T|\mathcal{F}_t)\frac{1-\mathrm{E}_t[\delta(\tau)]-\{1-\mathrm{E}_t[\delta(\tau)]\}\,Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}}{1-\{1-\mathrm{E}_t[\delta(\tau)]\}\,Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}} \\
&= f(t,T) + \mu(T|\mathcal{F}_t)\left\{1-\frac{\mathrm{E}_t[\delta(\tau)]}{1-\{1-\mathrm{E}_t[\delta(\tau)]\}\,Q\{\tau \le T \mid \tau > t, \mathcal{F}_t\}}\right\}. \\
&= f(t,T) + \mu(T|\mathcal{F}_t)\left\{1-\mathrm{E}_t[\delta(\tau)]\frac{P(t,T)}{V(t,T)}\right\}.
\end{aligned}
\qquad \square
$$

8.2 When the firm value follow a jump-diffusion process

Merton (1976) models a risky asset using a jump-diffusion process. The diffusion process represents the systematic risk and the jump process represents the nonsystematic risk or the firm-specific risk, which can be hedged. Merton (1976) assumes that the capital asset pricing model (CAPM) holds for equilibrium returns, thus the expected instantaneous return of the risky asset equals riskless rate r. The jump process does not affect the return of the asset but increases its volatility. Zhou (1997) uses a jump-diffusion process to model the firm value evolution. The jump process follows a compound Poisson process whose occurrence follows a Poisson process and the amplitude is Lognormally distributed. When default can occur only at maturity, Zhou (1997) obtains the closed-form solution for the probability of default. When default can occur continuously in time, the closed-form solution is not obtained. A Monte Carlo approach is used to obtain the default probability.

Chen and Panjer (2006) assume that the firm value evolution follows a jump-diffusion process whose jump process is a Poisson-Exponential compound Poisson process. Chen and Panjer (2006) assume default can occur continuously in time. Using the property of memoryless property of the Exponential distribution, Chen and Panjer (2006) obtain the closed-form solution for the mean recovery rate and obtain the implied default probability from the credit spread. In this section, we show the forward intensity of the jump-diffusion model of Chen and Panjer (2006) and link it to credit spread. We assume the firm value evolution follows

$$
dV_t = rV_t dt + \sigma V_t dZ_t + (J_{N_t} dN_t - \lambda\mu_J dt)V_t,
$$

using the notation:
V_t: value of the firm at time t,
r: instantaneous expected rate of return,
σ^2: instantaneous variance of return,

Z_t: a standard Brownian motion,
N_t: total number of jumps of firm value up to time t,
J_{N_t}: jump size as a proportion of V_t of the N_t-th jump,

where $(N_t)_{t\geq 0}$ is a counting process which follows a Poisson process with parameter λ and is independent of Z_t. The jump amplitudes $J_1, J_2, \cdots$ are assumed to be independent of $(N_t)_{t\geq 0}$, be independently and identically distributed, and whose moment generating function exists with mean μ_J and variance σ_J^2. The instantaneous return of the value of a firm is $\mathrm{E}\left[\frac{dV_t}{V_t}\right] = rdt$. Based on Itô's lemma, we have

$$d\ln V_t = \left(r - \frac{\sigma^2}{2} - \lambda\mu_J\right) dt + \sigma dZ_t + \ln(J_{N_t}+1)dN_t.$$

The percentage change $\frac{dV_t}{V_t}$ is bounded below by -1 and can go up to infinity. The range of the jump amplitude J_i thus can only take values in $[-1, \infty)$. Assume that $Y_i = J_i + 1$ for $i = 1 \cdots N_t$. Then Y_i is an nonnegative impulse function which takes value 1 when there is no jump and takes values from 0 to ∞ other than 1 if there is a jump. Let $Y(t) = 1$ when $N_t = 0$ and

$$Y(t) = \prod_{i=1}^{N_t} Y_i = \prod_{i=1}^{N_t} (J_i + 1),$$

when $N_t \neq 0$. The firm value at time t equals

$$V_t = V_0 \exp\left[\left(r - \frac{\sigma^2}{2} - \lambda\mu_J\right) t + \sigma Z_t\right] Y(t).$$

Default occurs once the firm value hits the default threshold D. When the firm value is away from the threshold D, default can occur only caused by a jump. When the firm value is approaching D at a value of $D+$, the firm value will diffuse to D with probability one. We say default is caused by diffusion if for every $\epsilon > 0$, there is a $\delta > 0$ such that $U(\tau - \delta) - U(\tau) < \epsilon$. Otherwise, we say default is caused by a jump. Therefore, $\{\tau = t\} = \{L[0,t) > D, V_t \leq D\} = \{L[0,t) > D, V_{t-} = D+, V_t \leq D\} \cup \{L[0,t) > D, V_{t-} > D, V_t \leq D\}$. For a jump-diffusion model, let τ_d denote the time of default caused by diffusion and τ_j denote the time of default caused by a jump. Let the cause of the default event be

$$J = \begin{cases} d, & \tau = \tau_d \\ j, & \tau = \tau_j \end{cases}.$$

We do not consider the transaction and liquidation cost in this jump-diffusion model. If default is caused by diffusion, the recovery rate $\delta(\tau_d)$ is 100%. If default is caused by a jump, the recovery rate $\delta(\tau_j)$ is less than or equal to 100%. The forward default intensity caused by diffusion, conditioned on future survival, is denoted by $\mu^{(d)}(T|\mathcal{F}_t)$. The forward default intensity caused by a jump, conditioned on future survival, is denoted by $\mu^{(j)}(T|\mathcal{F}_t)$. Chen and Panjer (2006) show that the credit spread follows the following theorem.

Theorem 8.2 Chen and Panjer (2006)

$$\begin{aligned} & R(t,T) - r(t,T) \\ = \;& -\frac{1}{T-t} \ln\left(1 - \{1 - E_t[\delta(\tau_s) \mid t < \tau_s \le T]\} Q\{\tau \le T, J = j | \tau > t, \mathcal{F}_t\}\right). \end{aligned} \tag{14}$$

An equivalent equation to equation (14), in the form of bond prices, is given by

$$V(t,T) = P(t,T)\left(1 - \{1 - \mathrm{E}_t[\delta(\tau_s) \mid t < \tau_s \le T]\} Q\{\tau \le T, J = j | \tau > t, \mathcal{F}_t\}\right). \tag{15}$$

Chen and Panjer (2006) assume that $X = -\ln Y_i$ is Exponentially distributed with probability density function

$$f_X(x) = \alpha e^{-\alpha x},\ \alpha > 0.$$

This implies that $f_J(x) = \alpha(x+1)^{\alpha-1},\ -1 < x < 0$. Chen and Panjer (2006) find that

$$f_{\delta(\tau_s)}(x) = \alpha x^{\alpha-1} \quad 0 < x < 1,$$

which is a beta distribution with parameters $(\alpha, 1)$ or a power distribution. The mean recovery rate for default caused by a jump is

$$\mathrm{E}_t[\delta(\tau_s) \mid t < \tau_s \le T] \;=\; \frac{\alpha}{\alpha+1}.$$

The recovery rate for default caused by a jump is not a function a maturity T. The forward spread follows

Theorem 8.3 *When $X = -\ln(J+1)$ is Exponentially distributed with pdf $f_X(x) = \alpha e^{-\alpha x},\ \alpha, x > 0$, the forward spread follows*

$$F(t,T) - f(t,T) \;=\; \frac{\mu^{(j)}(T|\mathcal{F}_t)}{\alpha+1} e^{-\int_t^T \mu(z|\mathcal{F}_t)dz} \frac{P(t,T)}{V(t,T)}. \tag{16}$$

Proof:

Taking the derivative of the log value of equation (15), we have

$$\begin{aligned} F(t,T) &= -\frac{\partial}{\partial T} \ln V(t,T) \\ &= -\frac{\partial}{\partial T} \ln\left\{P(t,T)\left(1 - \{1 - \mathrm{E}_t[\delta(\tau_s) \mid t < \tau_s \le T]\} Q\{\tau \le T, J = j | \tau > t, \mathcal{F}_t\}\right)\right\} \\ &= f(t,T) - \frac{\partial}{\partial T} \ln\left[\left(1 - \{1 - \mathrm{E}_t[\delta(\tau_s) \mid t < \tau_s \le T]\} Q\{\tau \le T, J = j | \tau > t, \mathcal{F}_t\}\right)\right] \\ &= f(t,T) - \frac{\frac{\partial}{\partial T}\left(1 - \{1 - \mathrm{E}_t[\delta(\tau_s) \mid t < \tau_s \le T]\} Q\{\tau \le T, J = j | \tau > t, \mathcal{F}_t\}\right)}{1 - \{1 - \mathrm{E}_t[\delta(\tau_s) \mid t < \tau_s \le T]\} Q\{\tau \le T, J = j | \tau > t, \mathcal{F}_t\}} \\ &= f(t,T) + \frac{\{1 - \mathrm{E}_t[\delta(\tau_s) \mid t < \tau_s \le T]\} \frac{\partial}{\partial T} Q\{\tau \le T, J = j | \tau > t, \mathcal{F}_t\}}{1 - \{1 - \mathrm{E}_t[\delta(\tau_s) \mid t < \tau_s \le T]\} Q\{\tau \le T, J = j | \tau > t, \mathcal{F}_t\}} \\ &= f(t,T) + \frac{\{1 - \mathrm{E}_t[\delta(\tau_s) \mid t < \tau_s \le T]\} e^{-\int_t^T \mu(z|\mathcal{F}_t)dz} \mu^{(j)}(T|\mathcal{F}_t)}{1 - \{1 - \mathrm{E}_t[\delta(\tau_s) \mid t < \tau_s \le T]\} Q\{\tau \le T, J = j | \tau > t, \mathcal{F}_t\}} \\ &= f(t,T) + \mu^{(j)}(T|\mathcal{F}_t) \frac{\frac{1}{\alpha+1} e^{-\int_t^T \mu(z|\mathcal{F}_t)dz}}{V(t,T)/P(t,T)} \\ &= f(t,T) + \frac{\mu^{(j)}(T|\mathcal{F}_t)}{\alpha+1} e^{-\int_t^T \mu(z|\mathcal{F}_t)dz} \frac{P(t,T)}{V(t,T)}. \qquad \square \end{aligned}$$

9 Conclusion

In this paper, we discuss the existence of default intensities for the first passage model when default can occur only if default has not occurred in the past. The expected rate of default occurrence given the firm has not defaulted with information $\mathcal{F}_t$, $t \geq 0$, is $\mu(t \mid \mathcal{F}_t) = \lim_{h\downarrow 0} \frac{\mathrm{E}[N(t+h)-N(t)|\tau>t,\mathcal{F}_t]}{h}$. In this paper, it is referred to as the instantaneous default intensity conditioned on survival. It is a time-dependent function when the firm value is observable and is a stochastic process when the firm value is not observable. Its relationship to the conditional hazard rate is given by $\mu(t \mid \mathcal{F}_t) = h(t|t)$. Without conditioned on survival, the expected rate of default occurrence given information up to current time $t \geq 0$, $\lambda(t)$, is the product of the indicator function $1_{\{\tau \leq t\}}$ and $\mu(t|\mathcal{F}_t)$. $\lambda(t)$ is referred to as the instantaneous default intensity. When a default event is a recurrent event, then default occurrence is no more conditioned on survival.

For the firm value process stated in section 2, the instantaneous default intensity exists only when the stopping time is totally inaccessible. For a diffusion process or a jump-diffusion process, the probability that a sequence of stopping approach the default time τ strictly from below is not zero. Therefore, the instantaneous default intensity does not exist in these two types of firm value processes.

We point it out that the probability of default conditioned on information up to current time, should not be calculated based on the instantaneous default intensity. It should be calculated based on the forward intensity. The forward expected rate of default occurrence at time $T > t$, given default has not occurred by time T, is defined by $\mu(T \mid \mathcal{F}_t) = \lim_{h\downarrow 0} \frac{\mathrm{E}[N(T+h)-N(T)|\tau>T,\mathcal{F}_t]}{h}$, $0 \leq t < T < \infty$. We call it the forward default intensity conditioned on future survival. The relationship between this forward default intensity and the condition hazard rate is given by $\mu(T \mid \mathcal{F}_t) = h(T|t)$. The $\lambda(T|t) = \lim_{h\downarrow 0} \frac{\mathrm{E}[N(T+h)-N(T)|\tau>t,\mathcal{F}_t]}{h}$, $0 \leq t < T < \infty$, is called the forward default intensity conditioned on current survival. Its relationship with $\mu(T|\mathcal{F}_t)$ is given by $\lambda(T|t) = 1_{\{\tau>T|\tau>t,\mathcal{F}_t\}}\mu(T|\mathcal{F}_t)$. When $f_\tau(T|t)$ and $S_\tau(T|t)$ can be found, their ratio equal $\mu(T|\mathcal{F}_t)$. The forward default intensity does possibly exist not only for the totally inaccessible stopping times, but also for accessible stopping times and stopping times neither accessible nor totally inaccessible.

The yield spread and the forward spread are both obtained in terms of forward intensities in equation (13), (14), (15) and (16). When the firm value follows a jump-diffusion process, the forward default intensity can be separated as the sum of the forward default intensity caused by diffusion and the forward default caused by a jump. Therefore, the structural model does define an intensity process, a forward one but not always the instantaneous one.

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