Pricing Credit Card Loans and Credit Card Asset Backed Securities with Default Risks

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Abstract

In this paper we extend the Jarrow and Deventer (1998) model to allow for considering default risks for valuing the credit card loans and credit card asset-backed securities. We derive closed-form solutions in a continuous-time framework, and provide a numerical method to value credit card asset-backed securities in a discrete-time framework as well. We also use the market segmentation argument to describe the characteristics of the credit card industry. From our simulation results, we find that the shapes of forward-rate term structure and the forward spread (default risk premium) play most important roles in determining the value of credit-card loans and credit-card asset-backed securities.

Key words：

Credit-card loans, Credit-card asset backed securities, Default risks.

This Version: March, 2006

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1. Introduction

The average annual growth rate of consumer credit (93% of which is in the form of credit card receivables) was over 12% between 1980 and 2002 according to the 2003 Federal Reserve statistics reports. Before 1987, growth rate is on average upwards of 15%. After 1987, securitization became integral to credit card industry growth. Citicorp led the sector through the capital crunch of the early 1990s, increasing its credit card accounts 42% between 1990 and 1992, and stimulated growth to 18% in 1994 and 22% in 1995. By 1996, securitized credit card receivables exceeded $180 billion, at which time credit card comprised 48.4% of the non-mortage ABS market. By 2001, credit card securitization had grown to $339.1 billion (see Calomris and Mason (2004)).

As mentioned earlier, the market of credit card loans grows fast. However, it is difficult to value credit card loans since they pay/charge rates which differ from market rates on equivalent-risk financial securities. This major difference has been attributed to markets with imperfect competition, probably due to either market frictions, regulatory barrier, or adverse selection problems under asymmetric information (see for example, Hutchison and Pennacch's (1996)). Another important feature of credit card loan is that these loans have high default risks. Observing from the market, the default rates for credit card loan are on average much higher than other loans. Hence it is important to capture the characteristics of credit card loans, including interest rate differential and default risks when one constructs a model to value these products.

Apart from Jarrow and Deventer (1998) model, the existing literature regarding pricing credit card loan and credit card backed securities is rare. In this paper we use the arbitrage-free price method to construct a discrete-time model and a continuous model which can capture the characteristics of interest rate differential and default risks for pricing risky credit card loan assets, respectively. To do so, we first modify the Jarrow and Deventoer's model by extending the Heath-Jarrow-Morton (1990) term-structure model to allow for considering default risks. Further, we use the market segmentation argument to describe the characteristics of interest rate differential observed in the credit card industry.

Our model has three features. First, it takes existing spreads as an input into the model rather than deriving the model from implications on default probabilities and recovery rate. Second, rather than work with spot yield curves for default-free and risky debt, we work with "forward rates" and "forward spreads". The advantages of using forward rates are that the current term structure is an input to the model and the forward rates can describe short rates while the short rates cannot describe forward rates. Third, in one hand, we provide a closed-form solution for valuing credit card loans and its securitization products in a continuous-time framework. In the other hand, we offer an applicable lattice approach to value the credit card loan related products in a discrete-time framework.

From our simulation results, we find that the shapes of forward-rate term structure and the forward spread (default risk premium) play most important roles in determining the value of credit-card loans and credit-card asset-backed securities. To our best knowledge, our model is the first one to investigate how .the parameters of default risks affect the value of credit card loans and credit card asset backed securities.

This paper is composed of the following sections. Apart from introduction, section 2 briefly reviews the literature for pricing credit card loan and credit card loan ABS. Section 3 constructs a discrete-time model for pricing credit card loan assets with default risks. Section 4 derives a closed-form solution for valuing credit card loan related securities in a continuous-time framework. Section 5 sets up an applicable lattice approach to value the credit card loan related products in a discrete-time framework. Section 6 uses numerical examples to investigate how the key parameters in our model affect the values of credit card loans.. Finally we draw conclusions in section 7.

2 Literature Review

2.1 The pricing models for credit card loans

The existing literature on valuing credit card loans includes Ausubel (1991), O'Brien et al. (1994), the Office of Thrift Supervision (1994), Hutchison and Pennacchi (1996), and Jarrow and Deventer (1998). Ausubel (1991) and the Office of

Thrift Supervision (1994) compute present values by using a model with deterministic credit card loan growth, rates paid, and interest rates. This way trivializes the problem, because the interest rate risk and stochastic growth are the major confounding factors in determining present values. The Office of Thrift Supervision (1994) measures the interest rate risk of credit card loan balances by computing their "duration". Mixing deterministic and stochastic interest rate analysis in this manner only generates nonsensical results.

O'Brien et al. (1994) compute present values and interest rate sensitivities in two ways. One is done by the Office of Thrift Supervision, and the other is done by the discounted expected value using stochastic credit card loan balances, credit card loan rates, and interest rates. In the latter case, the expectation represents a present value only if investors are risk-neutral. Expectations are computed using a Monte Carlo simulation under this risk-neutrality assumption. Hutchison and Pennacchi (1996) calculate present values using an equilibrium-based model in an economy where interest rates follow a square root, mean-reverting process. Jarrow and Deventer (1998) provide an arbitrage-free procedure for computing present values in a stochastic interest rate environment using the Heath et al. (1992) methodology.

2.2 The Pricing Models for Credit Card ABS

The existing literature regarding credit card ABS focuses on the topic of introducing this merchandise and the structure of securitization, and the pricing model is rare comparably. Rosenthal (1988) discusses the key points of issuing credit card ABS with a real example. The repayment method of credit card ABS differs from others. Most studies in the literature discuss Bullet Amortization and Controlled Amortization, and Bhattacharya (1996) introduces the Pass Through method. Fabozzi (1997) offers the default model of credit card loans and extends it to the rating of credit card ABS. Morris (1990) provides the whole concept of the process of issuing credit card ABS. Dean (1999) finds that credit card receivables have some characteristics which affect the structure of ABS, such as the average refund period is shorter than others, they are not assured, and the default rate is high. He also discusses the differences and the advantages of Stand-Alone Trust and Master Trust.

3. THE MODEL

3.1 The framework of our model

In this paper we construct a framework for pricing credit card loans with default risks. Utilizing the risk-neutral pricing methodology, we develop an arbitrage-free model for valuing credit card loans. Credit card loans are difficult to value because the charge rate differs from the market rates on financial securities of equal risk. In order to describe this characteristic well, we use the market segmentation argument. There are numerous providers of credit cards and no major barriers to entry. Such a market structure leads to competitive performance, whereby prices adjust with costs and issuers earn a normal rate of profit. However, we can find that the credit card interest rate rather than other rate is inactive and the largest issuers fix their rates at 18-20 percent. According to the paper by Paul S. and Loretta J. (1995), there is imperfect competition in the credit card industry due to search costs, switch costs, and adverse selection.

The market segmentation hypothesis is that there are two types of traders, banks or financial institutions and individuals. The partition between these two types is based on their ability to issue credit cards. We assume that there are significant regulatory restrictions and entry or mobility barriers associated with credit card loans. Only banks or financial institutions, and not individual investors, can issue credit cards.

3.2 Valuation of credit card loans with default risks

Consider an economy on a finite time interval $[0,T^*]$. Periods are taken to be of length h > 0. Thus, a typical time-point t has the form $k*h$ for some integer k. At time t, $t=k*h$. It is assumed that at all time t, a full range of default-free zero-coupon bonds is trades, as does a full range of risky zero-coupon bonds. It is also assumed that the markets are free of arbitrage, and so there exists an equivalent martingale measure Q.

For any given pair of time points (t,T) with $0 \le t \le T \le T^* - h$, let $L(t)$ denotes the volume of credit card loans to a particular bank at time t, and $c(t)$ denotes the credit card loans' interest rate at time t.

The cash flow of credit card loans is as follows：

Table 1: The cash-flow of credit card loans

t	$t+h$	$t+2h$	---	$T-h$	T
$-L(t)$	$+L(t)*\exp\{c(t)\}$	$+L(t+h)*\exp\{c(t+h)\}$	---	$+L(T-2h)*\exp\{c(T-2h)\}$	$+L(T-h)*\exp\{c(T-h)\}$
	$-L(t+h)$	$-L(t+2h)$	---	$-L(T-h)$	

The cash flow of credit card loans shows no risks, but actually there are default risks. Under the risk-neutral measure, the expected risky cash flow discounted at riskless rates must be equal to the value of expected riskless cash flow discounted at risky discount rates. Hence, we use the risky discount rate to calculate the net present value of $L(t)$, and let $V_L(t)$ denote the net present value of $L(t)$ at time t to the bank.

For any given pair of time points (t,T), let $f(t,T)$ denote the forward rate on the default-free bonds applicable to the period $(T,T+h)$ and $r(t)$ denotes the short rate, and $r(t) = f(t,t)$. That is to say, $f(t,T)$ is the rate as viewed from time t for a default-free lending or investment transaction over the interval of $(T,T+h)$. The forward rate curve is assumed to evolve according to the process：

$$f(t+h,T) = f(t,T) + \alpha(t,T)h + \sigma_f(t,T)X_1\sqrt{h} \tag{1}$$

where $\alpha(t,T)$ and $\sigma_f(t,T)$ are the drift term and the volatility of the forward rate, respectively； X_1 is a random variable.

Let $\varphi(t,T)$ be the forward rate on the risky bonds implied from the spot yield curve, and $s(t,T)$ is the forward spread on the risky bonds and is defined as：

$$s(t,T) = \varphi(t,T) - f(t,T) \tag{2}$$

Assume that the forward spread follows the process given in equation (3).

$$s(t+h,T)=s(t,T)+\beta(t,T)h+\sigma_{spread}(t,T)X_2\sqrt{h} \quad (3)$$

where $\beta(t,T)$ and $\sigma_{spread}(t,T)$ are the drift term and volatility of the forward spread, respectively； X_2 is a random variable.

Under the risk-neutral measure, the present value of expected riskless cash flow must be discounted at risky discount rates. The cash flow includes default risks. Using $\varphi(t,T)$ as the discount rate, we can get the net present value of credit card loans at time t, $V_L(t)$, by equation (2).

The net present value of credit card loans at time t, $V_L(t)$, to the financial institution is as follows (the derivation of equation (4) can refer to Appendix A)：

$$V_L(t)=E_t[-L(t)+J(t)*\sum_{k=t/h}^{T/h-1}\frac{L(kh)*\exp\{c(kh)*h\}-L(kh+h)}{J(kh+h)}+\frac{J(t)*L(T)}{J(T)}] \quad (4)$$

Where $J(t)$ denotes the time t value of an account that uses an initial investment \$1 ($J(0)=1$), and rolls the proceeds over at the rate φ. That is to say,

$$J(t)=\exp\{\sum_{k=0}^{t/h-1}\varphi(kh,kh)*h\} \quad (5)$$

The value of credit card loan assets to the financial institution at time t is denoted by $C_L(t)$, and this equals the initial credit card loans plus their net present value. In other words,

$$C_L(t)\equiv L(t)+V_L(t) \quad (6)$$

3.3 Identifying the Risk-Neutral Drifts

In this section we derive recursive expressions for the drifts α and β of the forward rate and forward spread processes, respectively, in terms of their volatilities, σ_f and σ_{spread}.

First, denote $B(t)$ to be the time t value of a money-market account that uses an initial investment of \$1, and roll the proceeds over at the default-free short rate；that is,

$$B(t) = \exp\{\sum_{k=0}^{t/h-1} r(kh) * h\} \tag{7}$$

Let $Z(t,T)$ denote the price of a default-free bond discounted using $B(t)$. Under Q (martingale measure), all asset prices in the economy discounted by $B(t)$ will be martingales.

$$Z(t,T) = \frac{P(t,T)}{B(t)} \tag{8}$$

Since Z is a martingale under Q, we can get that

$$Z(t,T) = E^t[Z(t+h,T)] \tag{9}$$

or

$$E^t[\frac{Z(t+h,T)}{Z(t,T)}] = 1 \tag{10}$$

Under these assumptions, we can get that

$$\sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} \alpha(t,kh) * h^2 = \ln\{E^t[\exp\{-\sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} \sigma_f(t,kh) X_1 * h^{3/2}\}]\} \tag{11}$$

and

$$\sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} [\alpha(t,kh) + \beta(t,kh)] h^2 = \ln\{E^t[\exp\{-h^{3/2} \sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} [\sigma_f(t,kh) X_1 + \sigma_{spread}(t,kh) X_2]\}]\} \tag{12}$$

Using these two equations above, we obtain α and β in terms of σ_f and σ_{spread}. Under the Heath-Jarrow-Morton term-structure model, we can use the forward rate volatility and forward spread volatility to describe the drift terms of forward rate and forward spread. This method can decrease the inputs and simplify the whole model.

4. A Continuous-time Model

This section considers the continuous time economy with trading horizon [0, τ]. We first redefine the notations of the last section. Let f(t,T) is the instantaneous forward rate at time t for a default-free transaction at time T. φ (t,T) denotes the instantaneous forward rate on the risky bonds with maturity T. The instantaneous forward spread s(t,T) on the risky bonds is defined as the equation (2). The forward rate curve process and the forward spread process are assumed to follow the processes

$$df(t,T) = \alpha(t,T)dt + \sigma_f(t,T)dW_r(t)\,, \tag{13}$$

$$ds(t,T) = \beta(t,T)dt + \sigma_s(t,T)dW_s(t)\,, \tag{14}$$

where $\alpha(t,T)$, $\beta(t,T)$ are the drift terms and $\sigma_f(t,T)$, $\sigma_s(t,T)$ are the volatility coefficients. (W_r(t), W_s(t)) is a two-dimensional Brownian motion with instantaneous correlation ρ, and where $-1 \leqq \rho \leqq 1$. In order to price the credit card loan in the continuous time economy, we rewrite equation (4) and evaluate it at time 0[1].

$$V_L(0) = E_0\left[\sum_{i=0}^{T/h-1}\frac{L(ih)\left[\exp\left(c(ih)h\right)-\exp\left(\varphi(ih,ih)h\right)\right]}{\exp\left(\sum_{j=0}^{i}\varphi(jh,jh)h\right)}\right] \tag{15}$$

By analogy with equation (15), the net present value of the credit card loan at time 0 is given by

$$V_L(0) = E_0\left[\int_0^{\tau}\frac{L(t)\left[\exp\left(c(t)\right)-\exp\left(\varphi(t,t)\right)\right]}{\exp\left(\int_0^{t}\varphi(u,u)du\right)}dt\right]. \tag{16}$$

To obtain a closed-form solution for equation (16), we follow Jarrow and Deventer (1998) and consider the stochastic process for L(t) and c(t) as follows:

$$d\log L(t) = \left[\alpha_0 + \alpha_1 t + \alpha_2 r(t)\right]dt + \alpha_3 dr(t), \tag{17}$$

$$dc(t) = \left[\beta_0 + \beta_1 r(t)\right]dt + \beta_2 dr(t). \tag{18}$$

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1 The derivation of equation (16) can refer to Appendix B.

The solutions for the differential equations of (17) and (18) are presented as follows:

$$L(t)=L(0)\exp\left[\alpha_0 t+\alpha_1 t^2/2+\alpha_2\int_0^t r(u)du+\alpha_3\left(r(t)-r(0)\right)\right], \tag{19}$$

$$c(t)=c(0)+\beta_0 t+\beta_1\int_0^t r(u)du+\beta_2\left(r(t)-r(0)\right). \tag{20}$$

Substituting equations (19) and (20) into equation (15), we gobtain

$$V_L(0)=E_0\left(\int_0^\tau \exp\left(-\int_0^t \varphi(u,u)du\right)\cdot L(0)\cdot\exp\left[\alpha_0 t+\alpha_1 t^2/2+\alpha_2\int_0^t r(u)du+\alpha_3\left(r(t)-r(0)\right)\right]\cdot\right.$$

$$\left.\left[\exp\left(c(0)+\beta_0 t+\beta_1\int_0^t r(u)du+\beta_2\left(r(t)-r(0)\right)\right)-\exp\left(r(t)+s(t)\right)\right]dt\right).$$

After simplifying the above expression, we can rewrite $V_L(0)$ as follows:

$$V_L(0)=L(0)\cdot\exp\left(c(0)-\left(\alpha_3+\beta_2\right)r(0)\right)\cdot\int_0^\tau \exp\left((\alpha_0+\beta_0)t+\alpha_1 t^2/2\right)\cdot$$

$$E_0\left(\exp\left((\alpha_2+\beta_1-1)\cdot\int_0^t r(u)du+(\alpha_3+\beta_2)r(t)-\int_0^t s(u)du\right)\right)dt$$

$$-\,L(0)\cdot\exp\left(-\alpha_3 r(0)\right)\cdot\int_0^\tau \exp\left(\alpha_0 t+\alpha_1 t^2/2\right)\cdot$$

$$E_0\left(\exp\left((\alpha_2-1)\cdot\int_0^t r(u)du+(\alpha_3+1)r(t)-\int_0^t s(u)du+s(t)\right)\right)dt$$

(21)

To get a closed-form solution for the value of $V_L(0)$, we consider the case of a Gaussian economy in which the process of spot rate and spot spread under risk-neutral probability measure are as follows:

$$dr(t)=a_r\left[\bar{r}(t)-r(t)\right]dt+\sigma_r d\widetilde{W}_r(t) \tag{22}$$

$$ds(t)=a_s\left[\bar{s}(t)-s(t)\right]dt+\sigma_s d\widetilde{W}_s(t), \tag{23}$$

where r(t) = f(t,t) is the instantaneous spot rate, s(t) = s(t,t) is the instantaneous spot spread, a_r and a_s are constants, σ_r (σ_s) is the volatility of the spot rate (the spot spread). Further, $\bar{r}(t)$ and $\bar{s}(t)$ are the deterministic functions to fit the initial forward rate curve {f(0,T), $0\leqq T\leqq\tau$ } and forward spread curve {s(0,T), $0\leqq T\leqq\tau$ }. In order to avoid arbitrage and to match the initial curve, $\bar{r}(t)$ and $\bar{s}(t)$ must satisfy the following condictions:

$$\overline{r}(t) = f(0,t) + \left[\partial f(0,t)/\partial t + \sigma_r^2(1-e^{-2a_r t})/2a_r\right]/a_r \tag{24}$$

$$\overline{s}(t) = s(0,t) + \left[\partial s(0,t)/\partial t + \sigma_s^2(1-e^{-2a_s t})/2a_s\right]/a_s. \tag{25}$$

The solutions for equations (24) and (25) are then obtained as follows:

$$r(t) = f(0,t) + \sigma_r^2(e^{-a_r t}-1)^2/(2a_r^2) + \int_0^t \sigma_r e^{-a_r(t-u)} d\widetilde{W}_r(u) \tag{26}$$

$$s(t) = s(0,t) + \sigma_s^2(e^{-a_s t}-1)^2/(2a_s^2) + \int_0^t \sigma_s e^{-a_s(t-u)} d\widetilde{W}_s(u). \tag{27}$$

Let $X \equiv \begin{bmatrix} \int_0^t r(u)du \\ r(t) \\ \int_0^t s(u)du \\ s(t) \end{bmatrix}$, $\mu \equiv \begin{bmatrix} \mu_1(t) \\ \mu_2(t) \\ \mu_3(t) \\ \mu_4(t) \end{bmatrix}$, $\Sigma \equiv \begin{bmatrix} \sigma_1^2(t) & \sigma_{12}(t) & \sigma_{13}(t) & \sigma_{14}(t) \\ \sigma_{21}(t) & \sigma_2^2(t) & \sigma_{23}(t) & \sigma_{24}(t) \\ \sigma_{31}(t) & \sigma_{32}(t) & \sigma_3^2(t) & \sigma_{34}(t) \\ \sigma_{41}(t) & \sigma_{42}(t) & \sigma_{43}(t) & \sigma_4^2(t) \end{bmatrix}$,

$$\gamma \equiv \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix},$$

where X is a vector of normal random variables with mean, μ, and covariance matrix, Σ. γ is a vector of constants. Using equations (21), (24), (25), and above definitions, we can obtain the closed-form solution for $V_L(0)$ as follows:

$$\begin{aligned} V_L(0) &= L(0)\exp\left(c(0)-\left(\alpha_3+\beta_2\right)r(0)\right)\int_0^\tau \exp\left((\alpha_0+\beta_0)t+\alpha_1 t^2/2\right)\cdot \\ &\quad M\left(t,\alpha_2+\beta_1-1,\alpha_3+\beta_2,-1,0\right)dt - L(0)\exp\left(-\alpha_3 r(0)\right)\int_0^\tau \exp\left(\alpha_0 t+\alpha_1 t^2/2\right)\cdot \\ &\quad M\left(t,\alpha_2-1,\alpha_3+1,-1,1\right)dt. \end{aligned} \tag{28}$$

where[2] $M(t,\gamma_1,\gamma_2,\gamma_3,\gamma_4) \equiv E_0\left(e^{\gamma^T X}\right) = \exp\left(\gamma^T\mu+\gamma^T\Sigma\gamma\right)$ is the moment generating function of the normal random vector X.

$$\mu_1(t) \equiv \int_0^t f(0,u)du + \int_0^t \left[\sigma_r^2\left(\exp\left(-a_r u\right)-1\right)^2/(2a_r^2)\right]du$$

$$\mu_2(t) \equiv f(0,t) + \sigma_r^2\left(\exp\left(-a_r t\right)-1\right)^2/(2a_r^2)$$

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[2] The derivations for these integrals are presented in the Appendix C.

$$\mu_3(t) \equiv \int_0^t s(0,u)du + \int_0^t \left[\sigma_s^2 \left(\exp\left(-a_s u\right) - 1 \right)^2 / (2a_s^2) \right] du$$

$$\mu_4(t) \equiv s(0,t) + \sigma_s^2 \left(\exp\left(-a_s t\right) - 1 \right)^2 / (2a_s^2)$$

$$\sigma_1^2(t) \equiv \int_0^t \left[\sigma_r^2 \left(1 - \exp\left(-a_r (t-u)\right) \right)^2 / a_r^2 \right] du$$

$$\sigma_2^2(t) \equiv \int_0^t \sigma_r^2 \exp\left(-2a_r (t-u)\right) du$$

$$\sigma_3^2(t) \equiv \int_0^t \left[\sigma_s^2 \left(1 - \exp\left(-a_s (t-u)\right) \right)^2 / a_s^2 \right] du$$

$$\sigma_4^2(t) \equiv \int_0^t \sigma_s^2 \exp\left(-2a_s (t-u)\right) du$$

$$\sigma_{12}(t) \equiv \left(\sigma_r^2 / (2a_r^2) \right) \left(1 - \exp\left(-a_r t\right)^2 \right)$$

$$\sigma_{13}(t) \equiv \int_0^t (\sigma_r \sigma_s \rho) \left[1 - \exp\left(-a_r (t-u)\right) \right] \left[1 - \exp\left(-a_s (t-u)\right) \right] / (a_r a_s) du$$

$$\sigma_{14}(t) \equiv \int_0^t (\sigma_r \sigma_s \rho) \left[1 - \exp\left(-a_r (t-u)\right) \right] \exp\left(-a_s (t-u)\right) / a_r du$$

$$\sigma_{23}(t) \equiv \int_0^t (\sigma_r \sigma_s \rho) \left[1 - \exp\left(-a_s (t-u)\right) \right] \exp\left(-a_r (t-u)\right) / a_s du$$

$$\sigma_{24}(t) \equiv \int_0^t \sigma_r \sigma_s \rho \exp\left(-a_r (t-u) - a_s (t-u)\right) du$$

$$\sigma_{34}(t) \equiv \left(\sigma_s^2 / (2a_s^2) \right) \left(1 - \exp\left(-a_s t\right)^2 \right)$$

It is easily to show that the closed-form solution derived by Jarrow and Deventer (1998) is a special case of ours when the default risk parameters are set as zero. Hence we contribute the literature by adding the components of default risks of credit card loans which are an important characteristic observed in such loans.

5 Numerical Procedures

In this section we construct a lattice approach to value credit card loans. This lattice approach is easily implemented.

5.1 The Process

We describe the procedures of constructing the lattice as following contents.

5.1.1 Random variables

There are two random variables in the model above, X_1 and X_2.We assume that X_1 and X_2 are binominal random variables, and each variable respectively takes on the values $+1$ and -1 with probability $\frac{1}{2}$. Let ρ denote the correlation between these two variables and note that ρ may not be equal to zero or constant. It is also assumed that the joint distribution of (X_1, X_2) is

$$(X_1, X_2) = \begin{Bmatrix} (+1,+1), w.p.(1+\rho)/4 \\ (+1,-1), w.p.(1-\rho)/4 \\ (-1,+1), w.p.(1-\rho)/4 \\ (-1,-1), w.p.(1+\rho)/4 \end{Bmatrix} \tag{29}$$

5.1.2 The term structure of forward rate and forward rate volatility

A forward rate's term may be three types. One is downward sloping, another is upward sloping, and the other is flat. A number of different theories have been proposed. The simplest is the expectations theory, which conjectures that long-term interest rates should reflect expected future short-term interest rates. The segmentation theory conjectures that there need be no relationship among short-term, medium-term, and long-term interest rates. The short-term interest rate is determined by supply and demand in the short-term market；the medium-term interest rate is determined by supply and demand in the medium-term market；and so on. The liquidity preference theory argues that long-term interest rates should always be higher than short-term interest rates. The basic assumption underlying the theory is that investors prefer to preserve their liquidity. This leads to a situation that the shape of the curve is upward sloping.

For implementation reasons, we assume that forward rate volatility is of the form：

$$\sigma_f(t,T) = \sigma * \exp\{-\lambda(T-t)\} \tag{30}$$

where $\sigma > 0$ is a positive constant and $\lambda \geq 0$ is a non-negative constant.

This is a more realistic volatility structure for forward rates and is obtained by permitting volatility to depend on the forward rate's maturity, $(T-t)$. If $\lambda = 0$, then forward rate volatility is constant, $\sigma_f(t,T) = \sigma$. If $\lambda > 0$, then it implies that forward rate volatility increases as the maturity, $(T-t)$, decreases. This exponentially-dampened volatility structure exploits the fact that near-term forward rates are more volatile than distant forward rates.

5.1.3 The term structure of forward spread and forward spread volatility

According to the experiment by Zhou (2001), the term structure of credit spreads can generate various shapes, including upward-sloping, downward-sloping, flat, and hump-shaped. Hence, we can set the form of the term structure of forward spread with different types.

For implementation reasons, we also assume that the forward spread volatility is of the form：

$$\sigma_{spread}(t,T) = \sigma_s * \exp\{-\lambda_s (T-t)\} \tag{31}$$

where σ_s > 0 is a positive constant.

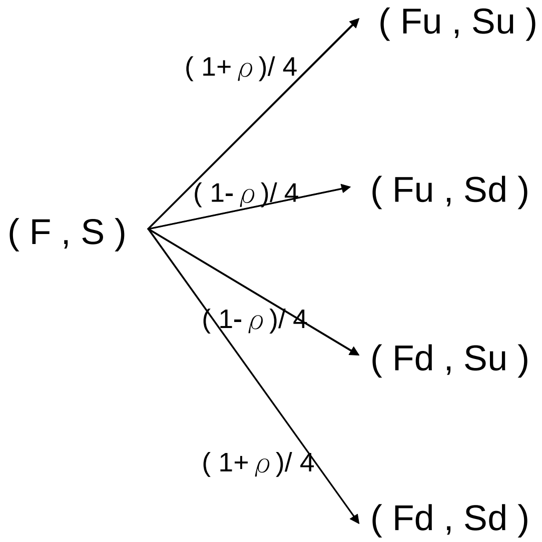
The term structure for forward spread is obtained by permitting volatility to depend on the forward spread's maturity, $(T-t)$. If $\lambda_s = 0$, then forward spread volatility is constant, $\sigma_{spread}(t,T) = \sigma_s$. If λ_s > 0, it implies that forward spread volatility increases as the maturity, $(T-t)$, decreases. By contrast, if λ_s < 0, it implies that forward spread volatility decreases as the maturity, $(T-t)$, decreases. That is to say, there are three possible shapes of forward spread volatility.

5.2 Implementation of our model

Going for implementation, we need the data of forward rate and forward spread. By those assumptions above, it may be easily implemented on a lattice. The double-binomial structure which is described above results in a branching process with four branches emanating from each node. We achieve the risk-neutral drifts, α and β, by forward rate volatility and forward spread volatility. Once the risk neutral drifts have been computed, the possible value of forward rates and forward spreads one period out are obtained.

The branching lattice appears as follows. Let F_u and F_d refer to the forward rates that result from F if X_1 equals $+1$ and -1, separately. Let S_u and S_d refer to the forward spreads that result from S if X_2 equals $+1$ and -1, separately. The probability of each branching depends on the joint distribution of (X_1, X_2) and is shown in equation (13).

Figure 1: The branching lattice



6. Numerical Results

6.1 A simple example

We implement a simple example to demonstrate our model. Consider an economy on a finite time interval $[0, 2]$. Periods are taken to be of length h, and $h = 0.5$ (half-year). The cash flow of credit card loans is as follows：

Table 2: The cash-flow of credit card loans for five periods

0	0.5	1	1.5	2
$-L(0)$	$+L(0)*\exp\{c(0)\}$	$+L(0.5)*\exp\{c(0.5)\}$	$+L(1)*\exp\{c(1)\}$	$+L(1.5)*\exp\{c(1.5)\}$
	$-L(0.5)$	$-L(1)$	$-L(1.5)$	

According to the whole model we describe above, we need some input for implementation. Depending on the revolving credit amounts in Taiwan shown in chapter 1, we set the volume of credit card loans, $L(t)$, increasing with a fixed rate, g, and the initial credit card loan amounts to NT$100 billion. The half-year growth rate, g, is 5%. According to the paper by Paul S. and Loretta J. (1995), we can find that the credit card interest rate is sticky, and thus we set up the credit card interest rate at time t, $c(t)$, to equal 19% all the time.

We can achieve the risk-neutral drifts, α and β, by forward rate volatility and forward spread volatility. Once the risk neutral drifts have been computed, the possible values of the forward rates and forward spreads one period out are obtained.

Assume that the forward rate's volatility, σ_f, follows equation (14), the volatility, σ, equals 2% and the volatility reduction factor, λ, equals 0.1. That is to say, forward rate volatility is

$$\sigma_f(t,T) = 0.02 * \exp\{-0.1(T-t)\} \tag{32}$$

And the term structure of forward rate volatility is shown in Table 4.

Table 3: The term structure of forward rate volatility

$(T-t)$	0	0.5	1.0	1.5	2.0
$\sigma_f\ (T-t)$	0.0200	0.0190246	0.0180967	0.0172142	0.0163746

We also set forward spread volatility, σ_{spread}, with the same form and it follows equation (15). The volatility, σ_s, equals 2% and the volatility reduction factor, λ_s, equals 0.1. That is to say, forward spread volatility is

$$\sigma_{spread}(t,T) = 0.02 * \exp\{-0.1(T-t)\} \tag{33}$$

The term structure of forward spread volatility is shown in Table 4.

Table 4: The term structure of forward spread volatility

$(T-t)$	0	0.5	1.0	1.5	2.0
$\sigma_{spread}\,(T-t)$	0.0200	0.0190246	0.0180967	0.0172142	0.0163746

Using Mathematica (Wolfram 1988), the term structure of forward rate and forward spread is as follows：

Table 5: The term structure of forward rate and forward spread

Period	T	$(0,T)$	$f(0,T)$	$s(0,T)$
0	0	(0 , 0)	0.05	0.008
1	0.5	(0 , 0.5)	0.06	0.010
2	1.0	(0 , 1.0)	0.07	0.015
3	1.5	(0 , 1.5)	0.08	0.020
4	2.0	(0 , 2.0)	0.09	0.022
ρ	-0.074			

To make use of the data in Table 6 and the term structure of forward rate volatility and forward spread's volatility, we achieve the double-binomial structure results in a branching lattice with four branches emanating from each node.

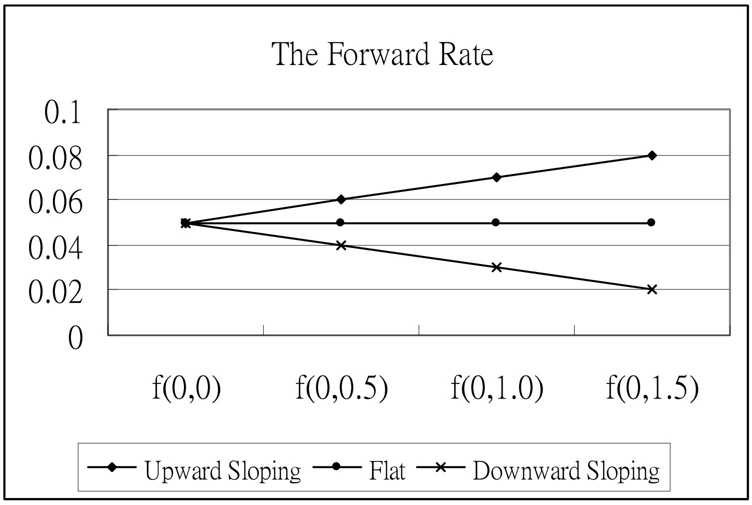
We now see that the net present value of credit card loans at time 0, $V_L(0)$, equals NT$23,071.80641 million, and the value of credit card loan assets to the financial institution at time 0, $C_L(0)$, equals NT$123,071.80641 million.

We next will go to the sensitivity analysis. In other words, we further discuss the major factors, forward rate, forward spread, forward rate volatility and forward spread volatility, in this model and how they affect the values of $V_L(0)$ and $C_L(0)$.

6.2 The effects of changes in the term structure of forward rate

The forward rate's term structure may be different shapes, including downward sloping, flat, and upward sloping. We calculate the net present value of credit card loans at time 0, $V_L(0)$, and the value of credit card loan assets, $C_L(0)$, under these three types.

Figure 2: Three types of term structure of forward rate



The results are shown in Table 7. We can clearly see that when the term structure of forward rate is upward sloping, the value of $C_L(0)$ is the smallest. When the term structure of forward rate is downward sloping, the value of $C_L(0)$ is the biggest.

We have already set the credit card interest rate to be fixed and equal to 19%. If the term structure of forward rate is upward sloping, it means that the capital cost in the future is larger than now. When the earnings rate is fixed, the value of $C_L(0)$ under this term structure will get the minimum value within these three situations.

Table 6: The value of $C_L(0)$ under three types of term structure of forward rate

The term structure of forward rate	Upward Sloping	Flat	Downward Sloping
$f(0,0)$	0.05	0.05	0.05
$f(0,0.5)$	0.06	0.05	0.04
$f(0,1.0)$	0.07	0.05	0.03
$f(0,1.5)$	0.08	0.05	0.02
$V_L(0)$	23071.80641	25891.17073	29808.70709
$C_L(0)$	123071.80641	125891.17073	129808.70709
Change %	---	2.29%	5.47%

6.3 The effects of changes in the term structure of forward spread

According to the experiment by Chunsheng Zhou (2001), the term structure of credit spreads may generate various shapes. We discuss here the effects of three basic types of forward spread on the net present value of credit card loans at time 0, $V_L(0)$, and the value of credit card loan assets, $C_L(0)$. We assume that the term structure of forward spread is as shown in Figure 6.

The results are shown in Table 8. The spread catches the default risks in credit card loans. We can clearly see that when the term structure of forward spread is upward sloping, the value of $C_L(0)$ is the smallest. When the term structure of forward spread is downward sloping, the value of $C_L(0)$ is the biggest. If the term structure of forward spread is upward sloping, it means that the default risks in the future are larger than now. It needs more premiums to compensate for the risks. But under the situation that the earning rate is fixed, the value of $C_L(0)$ will be the smallest.

Figure 2: Three types of term structure of forward spread

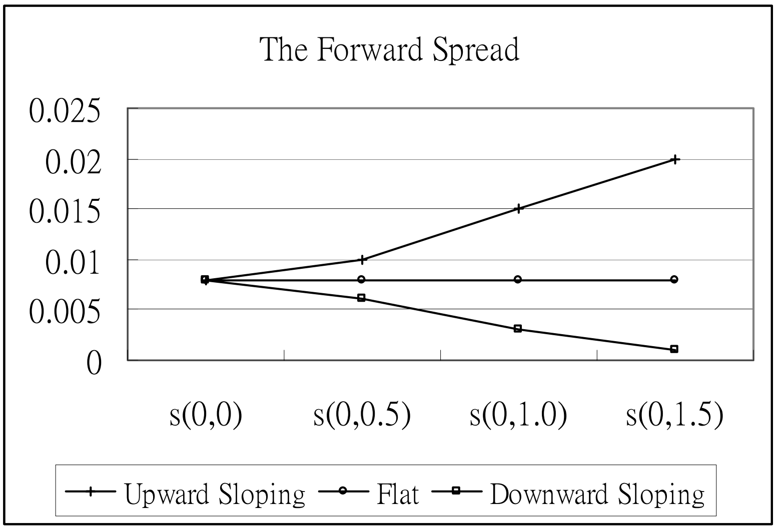


Table 7: The value of $C_L(0)$ under three types of term structure of forward spread

The term structure of forward spread	Upward Sloping	Flat	Downward Sloping
$s(0,0)$	0.008	0.008	0.008
$s(0,0.5)$	0.010	0.008	0.006
$s(0,1.0)$	0.015	0.008	0.003
$s(0,1.5)$	0.020	0.008	0.001
$V_L(0)$	23071.80641	23748.61399	24094.39675
$C_L(0)$	123071.80641	123748.61399	124094.39675
Change %	---	0.55%	0.83%

6.4 The effects of changes in forward rate volatility

As we mentioned in chapter 3, we assume that forward rate volatility abides by equation (14). According to the empirical performance of the single factor, constant volatility version of the interest rate contingent claims valuation model of Heath, Jarrow, and Morton (1992) by Bjorn Flesaker (1993), which uses a generalized method of moments (GMM) and tests on three years of daily data for Eurodollar futures and futures options, the result shows that the estimated volatility is extremely close to 0.02. The mean value equals 0.0200070, the minimum is 0.196846, and the maximum is 0.202845. Following the results of the empirical performance, we initially set forward rate's volatility, σ, to equal 2% and the volatility reduction factor, λ, to equal 0.1. We next see the effects of changes in forward rate volatility, and discuss in two topics, the volatility and volatility reduction factor.

6.4.1 The effects of volatility

We set the volatility, σ, to equal 2% and change the value to see the effects on the net present value of credit card loans at time 0, $V_L(0)$, and the value of credit card loan assets, $C_L(0)$. The results are shown in Table 8.

Table 8: The value of $C_L(0)$ with different volatility of forward rate volatility

$\sigma_f\ (T-t)$	σ = 1%	σ = 2%	σ = 3%	σ = 4%	σ = 5%
0	0.01	0.0200	0.03	0.04	0.05
0.5	0.009512	0.019025	0.028537	0.038049	0.047562
1.0	0.009048	0.018097	0.027145	0.036194	0.045242
1.5	0.008607	0.017214	0.025821	0.034428	0.043035
2.0	0.008187	0.016375	0.024562	0.032749	0.040937
$V_L(0)$	23072.5635	23071.80641	23070.6711	23068.8847	22958.45694
$C_L(0)$	123072.56335	123071.80641	123070.6711	123068.8847	122958.45694
Change‰	0.00615‰	---	-0.00923‰	-0.02374‰	-0.92100‰

We can find that when the volatility, σ, increases, the term structure of forward rate volatility, σ_f, decreases rapidly with the maturity, and the value of $C_L(0)$ decreases. This exponentially dampened volatility structure exploits the fact that near-term forward rates are more volatile than distance forward rates. When the volatility increases, it means that the forward rate is more uncertain and this makes the value of $C_L(0)$ smaller.

6.4.2 The effects of the volatility reduction factor

We also change the value of forward rate's volatility reduction factor and discuss the results, which are shown in Table 9.

Table 9: The value of $C_L(0)$ with different volatility reduction factors of forward rate volatility

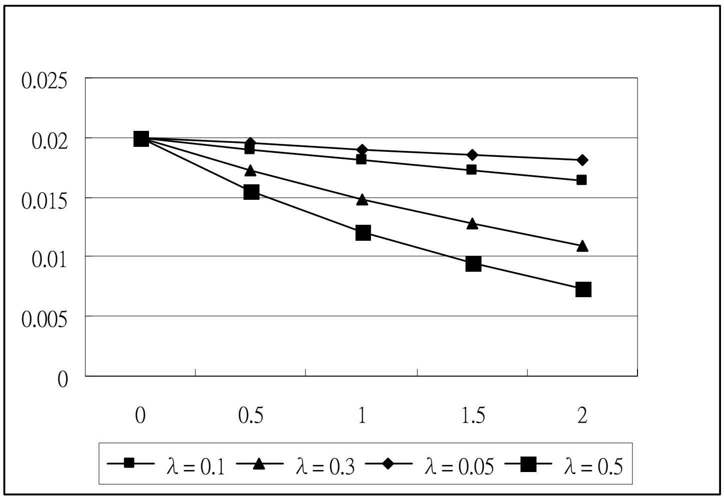
$\sigma_f\ (T-t)$	λ = 0.05	λ = 0.1	λ = 0.3	λ = 0.5
0	0.0200	0.0200	0.0200	0.0200
0.5	0.019506	0.0190246	0.017214	0.015576
1.0	0.019025	0.0180967	0.014816	0.012131
1.5	0.018554	0.0172142	0.012753	0.009447
2.0	0.018097	0.0163746	0.010976	0.007358
$V_L(0)$	23071.71402	23071.80641	23072.18038	23072.36517
$C_L(0)$	123071.71402	123071.80641	123072.18038	123072.36517
Change ‰	-0.00075‰	---	0.00304‰	0.00454‰

We find that when the volatility reduction factor, λ, increases, the slope of the term structure of forward rate volatility decreases. This makes the forward rate volatility be more stable and also makes the value of $C_L(0)$ increase.

6.5 The effects of changes in forward spread volatility

For implementation reasons, we assume that the term structure of forward spread volatility abides by equation (15). We first discuss the effects of the volatility reduction factor, λ_s, and set the value to be positive, zero, and negative. Please see Figure 3 and the results are shown in Table 10.

Figure 3: The term structure of forward rate volatility with different volatility reduction factors



We find that if $\lambda_s = 0$, then the term structure of forward spread volatility is flat； that is to say, $\sigma_{spread}(t,T) = \sigma_s$. If $\lambda_s > 0$, it implies that forward spread volatility decreases as the maturity, $(T-t)$, increases. By contrast, if $\lambda_s < 0$, it implies that forward spread volatility increases as the maturity, $(T-t)$, increases. When $\lambda_s < 0$, it makes forward spread volatility be larger than the others and leads to the value of $C_L(0)$ being the smallest.

Figure 4: The term structure of forward spread volatility with different volatility reduction factors

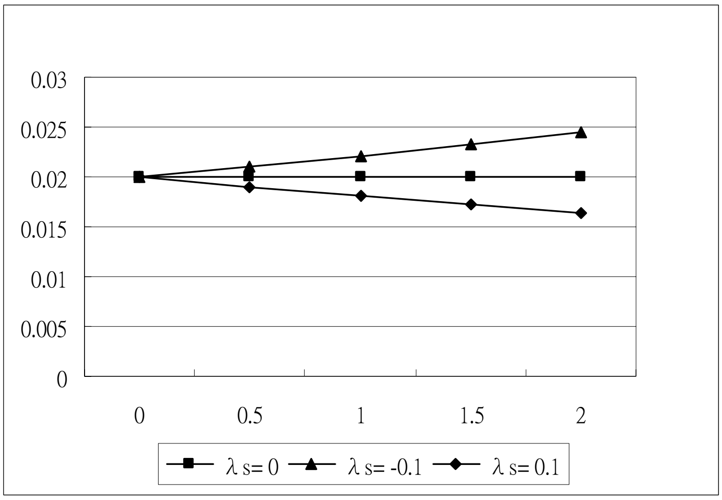


Table 10: The value of $C_L(0)$ under three types of volatility reduction factor of forward spread volatility

$\sigma_{spread}\ (T-t)$	λ_s = 0.1	λ_s = 0	λ_s = -0.1
0	0.0200	0.0200	0.0200
0.5	0.0190246	0.0200	0.0210254
1.0	0.0180967	0.0200	0.0221034
1.5	0.0172142	0.0200	0.0232367
2.0	0.0163746	0.0200	0.0244281
$V_L(0)$	23071.80641	22972.86488	22855.19066
$C_L(0)$	123071.80641	122972.86488	122855.19066
Change %	0.0805%	---	-0.0957%

6.5.1 The effects of volatility

We further discuss the effects of forward spread volatility under three types of λ_s.

I. If λ_s > 0, it implies that forward spread volatility decreases as the maturity, $(T-t)$, increases. We change the volatility, σ_s, and see the effects on the net present value of credit card loans at time 0, $V_L(0)$, and the value of credit card loan assets, $C_L(0)$. The results are shown in Table 12.

We find that when the volatility, σ_s, increases, the term structure of forward spread volatility , σ_{spread}, decreases rapidly with the maturity, and the value of $C_L(0)$ decreases. This exponentially dampened volatility structure exploits the fact that near-term forward spreads are more volatile than distance forward spreads. When the volatility increases, it means that the forward spread is more uncertain and makes the value of $C_L(0)$ smaller.

II. If $\lambda_s = 0$, it means that the term structure of forward spread volatility is flat； that is to say, $\sigma_{spread}(t,T) = \sigma_s$. We change the volatility, σ_s, and see the effects on the net present value of credit card loans at time 0, $V_L(0)$, and the value of credit card loan assets, $C_L(0)$. The results are shown in Table 13.

Table 11: The value of $C_L(0)$ with different volatility of forward spread volatility under $\lambda_s > 0$

$\sigma_{spread}\ (T-t)$	$\sigma_s = 1\%$	$\sigma_s = 2\%$	$\sigma_s = 3\%$	$\sigma_s = 4\%$	$\sigma_s = 5\%$
0	0.01	0.0200	0.03	0.04	0.05
0.5	0.009512	0.019025	0.028537	0.038049	0.047562
1.0	0.009048	0.018097	0.027145	0.036194	0.045242
1.5	0.008607	0.017214	0.025821	0.034428	0.043035
2.0	0.008187	0.016375	0.024562	0.032749	0.040937
$V_L(0)$	23493.1088	23071.80641	22406.56576	21607.65253	20797.23996
$C_L(0)$	123493.1088	123071.80641	122406.56576	121607.65253	120797.23996
Change %	0.3423%	---	-0.5405%	-1.1897%	-1.8482%

Table 12: The value of $C_L(0)$ with different volatility of forward spread volatility under $\lambda_s = 0$

$\sigma_{spread}\ (T-t)$	$\sigma_s = 1\%$	$\sigma_s = 2\%$	$\sigma_s = 3\%$	$\sigma_s = 4\%$	$\sigma_s = 5\%$
0	0.01	0.02	0.03	0.04	0.05
0.5	0.01	0.02	0.03	0.04	0.05
1.0	0.01	0.02	0.03	0.04	0.05
1.5	0.01	0.02	0.03	0.04	0.05
2.0	0.01	0.02	0.03	0.04	0.05
$V_L(0)$	23484.8248	22972.86488	22213.80386	21329.84318	20452.46576
$C_L(0)$	123484.8248	122972.86488	122213.80386	121329.84318	120452.46576
Change %	0.41632%	---	-0.61726%	-1.33608%	-2.04956%

We find that when the volatility, σ_s, increases, the term structure of forward spread volatility , σ_{spread}, is still flat, and the value of $C_L(0)$ decreases. This volatility structure exploits the fact that near-term forward spread volatility is the same as the distance forward spread volatility. When the volatility increases, it means that the forward spread is more uncertain and this makes the value of $C_L(0)$ smaller.

III. If λ_s < 0, it implies that forward spread's volatility increases as the maturity, $(T-t)$, increases. We change the volatility, σ_s, and see the effects on the net present value of credit card loans at time 0, $V_L(0)$, and the value of credit card loan assets, $C_L(0)$. The results are shown in Table 14.

We find that when the volatility, σ_s, increases, the term structure of forward spread's volatility , σ_{spread}, increases rapidly with the maturity, and the value of $C_L(0)$ decreases. This exponentially upward volatility structure exploits the fact that near-term forward spreads are more stable than distance forward spreads. When the volatility increases, it means that the forward spread is more uncertain and makes the value of $C_L(0)$ smaller.

Table 13: The value of $C_L(0)$ with different volatility of forward spread volatility under $\lambda_s < 0$

$\sigma_{spread}(T-t)$	$\sigma_s = 1\%$	$\sigma_s = 2\%$	$\sigma_s = 3\%$	$\sigma_s = 4\%$	$\sigma_s = 5\%$
0	0.01	0.0200	0.03	0.04	0.05
0.5	0.010513	0.021025	0.031538	0.042051	0.052564
1.0	0.011052	0.022103	0.033155	0.044207	0.055259
1.5	0.011618	0.023237	0.034855	0.046473	0.058092
2.0	0.012214	0.024428	0.036642	0.048856	0.061070
$V_L(0)$	23465.8728	22855.19066	21974.86713	21013.82813	20060.53263
$C_L(0)$	123465.8728	122855.19066	121974.86713	121013.82813	120060.53263
Change %	0.49707%	---	-0.71656%	-1.49881%	-2.27476%

We change the volatility, σ_s, under three types of term structure of forward spread volatility. Comparing Table 11, Table 12, and Table 13, we find that when $\lambda_s < 0$, the value of $C_L(0)$ changes more than the others. This means that if the slope of the term structure of forward spread volatility is upward, then the changes of volatility affect the value of $C_L(0)$ more than the others.

6.5.2 The effects of the volatility reduction factor

I. Under the situation of $\lambda_s > 0$, we change the volatility reduction factor and see the effects on the net present value of credit card loans at time 0, $V_L(0)$, and the value of credit card loan assets, $C_L(0)$. Please see Figure 9 and the results are shown in Table 15.

Figure 5: The term structure of forward spread volatility with different volatility reduction factors under $\lambda_s > 0$

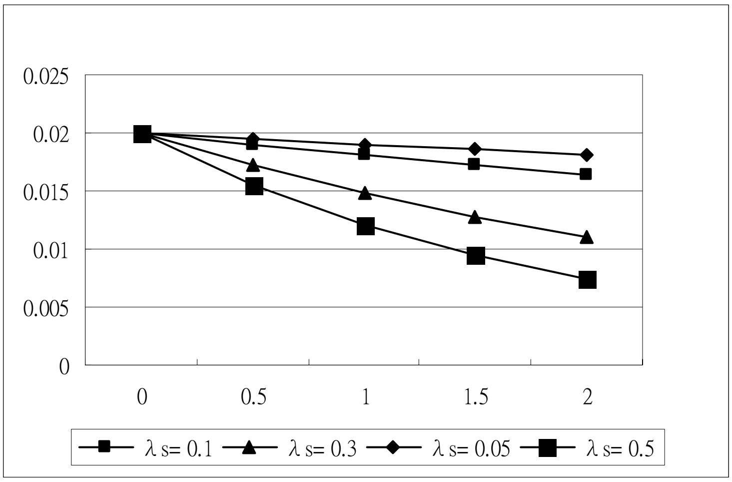


Table 14: The value of $C_L(0)$ with different volatility reduction factors of forward spread volatility under $\lambda_s > 0$

$\sigma_{spread}\ (T-t)$	$\lambda_s = 0.05$	$\lambda_s = 0.1$	$\lambda_s = 0.3$	$\lambda_s = 0.5$
0	0.0200	0.0200	0.0200	0.0200
0.5	0.019506	0.019025	0.017214	0.015576
1.0	0.019025	0.018097	0.014816	0.012131
1.5	0.018554	0.017214	0.012753	0.009447
2.0	0.018097	0.016375	0.010976	0.007358
$V_L(0)$	23023.53637	23071.80641	23240.47368	23354.93278
$C_L(0)$	123023.53637	123071.80641	123240.47368	123354.93278
Change %	-0.03922%	---	0.13705%	0.23005%

We find that when the volatility reduction factor, λ_s, increases, the slope of the term structure of forward spread volatility decreases. This makes forward spread volatility be more stable and also makes the value of $C_L(0)$ increase.

Figure 6: The term structure of forward spread volatility with different volatility reduction factors under $\lambda_s < 0$

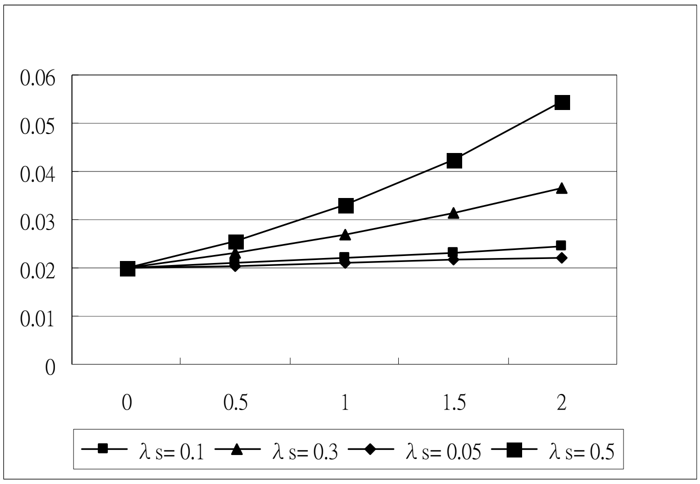


Table 15: The value of $C_L(0)$ with different volatility reduction factors of forward spread volatility under $\lambda_s < 0$

$\sigma_{spread}\ (T-t)$	$\lambda_s = 0.05$	$\lambda_s = 0.1$	$\lambda_s = 0.3$	$\lambda_s = 0.5$
0	0.0200	0.0200	0.0200	0.0200
0.5	0.020506	0.021025	0.023237	0.025681
1.0	0.021025	0.022103	0.026997	0.032974
1.5	0.021558	0.023237	0.031366	0.042340
2.0	0.022103	0.024428	0.036442	0.054366
$V_L(0)$	22919.78508	22855.19066	22524.63045	22052.6728
$C_L(0)$	122919.78508	122855.19066	122524.63045	122052.6728
Change %	0.05258%	---	-0.26906%	-0.65326%

II. Under the situation of $\lambda_s < 0$, we change the volatility reduction factor and see the effects on the net present value of credit card loans at time 0, $V_L(0)$, and the value of credit card loan assets, $C_L(0)$. Please see Figure 10 and the results are shown in Table 16.

We find that when the volatility reduction factor, λ_s, increases, the slope of the term structure of forward spread volatility increases. This makes the forward spread volatility be more volatile and also makes the value of $C_L(0)$ decrease.

Before closing this section, we should point out that it is straightforward to apply our pricing model to value credit card asset backed securities with default risk. The results for how the parameters of default risks affect the value of different trance in the credit card asset backed securities are pretty much similar to those of credit card loans. Therefore we do not report the results for pricing credit card asset backed securities for conciseness.

7. Conclusions

Using the risk-neutral pricing methodology, we provide an arbitrage-free model for valuing credit card loans and credit card loan ABS. The model is based on expanding the Heath-Jarrow-Morton (1990) term-structure model to allow for considering default risks. The credit card industry is imperfect competition and the interest rate is sticky all the time, and so we use the market segmentation argument to describe this characteristic.

Our model has three features. First, it takes existing spreads as an input into the model rather than deriving the model from implications on default probabilities and recovery rate. Second, rather than work with spot yield curves for default-free and risky debt, we work with “forward rates” and “forward spreads”. The advantages of using forward rates are that the current term structure is an input to the model and the forward rates can describe short rates while the short rates cannot describe forward rates. Third, in one hand, we provide a closed-form solution for valuing credit card loans and its securitization products in a continuous-time framework. In the other hand, we offer an applicable lattice approach to value the credit card loan related products in a discrete-time framework.

We find that the shapes of forward-rate term structure and the forward spread (default risk premium) play most important roles in determining the value of credit-card loans and credit-card asset-backed securities.

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Appendix A：The derivation of equation (4).

$$
\begin{aligned}
V_L(t) &= E_t[-L(t) + \frac{1}{\exp\{\varphi(t,t)*h\}}[L(t)*\exp\{c(t)*h\} - L(t+h)] \\
&+ \frac{1}{\exp\{[\varphi(t,t)+\varphi(t+h,t+h)]*h\}}[L(t+h)*\exp\{c(t+h)*h\} - L(t+2h)] \\
&+ \frac{1}{\exp\{[\varphi(t,t)+\varphi(t+h,t+h)+\varphi(t+2h,t+2h)]*h\}}[L(t+2h)*\exp\{c(t+2h)*h\} - L(t+3h)] \\
&+\cdots\cdots+\cdots\cdots+\cdots\cdots+ \frac{1}{\exp\{\sum_{i=\frac{t}{h}}^{\frac{T}{h}-2}\varphi(ih,ih)*h\}}[L(T-2h)*\exp\{c(T-2h)*h\} - L(T-h)] \\
&+ \frac{1}{\exp\{\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1}\varphi(ih,ih)*h\}}[L(T-h)*\exp\{c(T-h)*h\}]] \\
&= E_t[-L(t) + \frac{J(t)}{J(t+h)}[L(t)*\exp\{c(t)*h\} - L(t+h)] + \frac{J(t)}{J(t+2h)}[L(t+h)*\exp\{c(t+h)*h\} - L(t+2h)] \\
&+ \frac{J(t)}{J(t+3h)}[L(t+2h)*\exp\{c(t+2h)*h\} - L(t+3h)] + \cdots\cdots \\
&+ \frac{J(t)}{J(T-h)}[L(T-2h)*\exp\{c(T-2h)*h\} - L(T-h)] + \frac{J(t)}{J(T)}[L(T-h)*\exp\{c(T-h)*h\}]] \\
&= E_t[-L(t) \\
&+[\frac{J(t)}{J(t+h)}*L(t)*\exp\{c(t)*h\} + \frac{J(t)}{J(t+2h)}*L(t+h)*\exp\{c(t+h)*h\} + \frac{J(t)}{J(t+3h)}*L(t+2h)*\exp\{c(t+2h)*h\} \\
&+\cdots\cdots+ \frac{J(t)}{J(T-h)}*L(T-2h)*\exp\{c(T-2h)*h\} + \frac{J(t)}{J(T)}*L(T-h)*\exp\{c(T-h)*h\}] \\
&-[\frac{J(t)}{J(t+h)}*L(t+h) + \frac{J(t)}{J(t+2h)}*L(t+2h) + \frac{J(t)}{J(t+3h)}*L(t+3h) + \cdots\cdots + \frac{J(t)}{J(T-h)}*L(T-h)]] \\
&= E_t[-L(t) + J(t)[\sum_{k=t/h}^{T/h-1}\frac{L(kh)*\exp\{c(kh)*h\}}{J(kh+h)} - \sum_{k=t/h}^{T/h-2}\frac{L(kh+h)}{J(kh+h)}]] \\
&= E_t[-L(t) + J(t)[\sum_{k=t/h}^{T/h-1}\frac{L(kh)*\exp\{c(kh)*h\} - L(kh+h)}{J(kh+h)} + \frac{L(T)}{J(T)}]] \\
&= E_t[-L(t) + J(t)*\sum_{k=t/h}^{T/h-1}\frac{L(kh)*\exp\{c(kh)*h\} - L(kh+h)}{J(kh+h)} + \frac{J(t)*L(T)}{J(T)}]
\end{aligned}
$$

Appendix B: The derivation of equation (15)

From equation (4), the net present value of the credit card loan can be expressed as

$$V_L(0) = E_0\left[-L(0) + \sum_{i=0}^{T/h-2} \frac{L(ih)\exp\big(c(ih)h\big) - L((i+1)h)}{\exp\left(\sum_{j=0}^{i}\varphi(jh, jh)h\right)} + \frac{L(T-h)\exp\big(c(T-h)h\big)}{\exp\left(\sum_{j=0}^{T/h-1}\varphi(jh, jh)h\right)} \right]$$

$$= E_0\left[\sum_{i=0}^{T/h-2} \frac{L(ih)\exp\big(c(ih)h\big)}{\exp\left(\sum_{j=0}^{i}\varphi(jh, jh)h\right)} + \frac{L(T-h)\exp\big(c(T-h)h\big)}{\exp\left(\sum_{j=0}^{T/h-1}\varphi(jh, jh)h\right)} \right.$$

$$\left. - \frac{L(0)\exp\big(\varphi(0,0)h\big)}{\exp\big(\varphi(0,0)h\big)} - \sum_{i=0}^{T/h-2} \frac{L((i+1)h)\exp\Big(\varphi\big((i+1)h,(i+1)h\big)h\Big)}{\exp\left(\sum_{j=0}^{i}\varphi(jh, jh)h\right)\exp\Big(\varphi\big((i+1)h,(i+1)h\big)h\Big)} \right]$$

$$= E_0\left[\sum_{i=0}^{T/h-1} \frac{L(ih)\Big[\exp\big(c(ih)h\big) - \exp\big(\varphi(ih,ih)h\big)\Big]}{\exp\left(\sum_{j=0}^{i}\varphi(jh, jh)h\right)} \right].$$

Appendix C: Derivation of expression (28)

From equation (26), we have

$$r(t) = f(0,t) + \sigma_r^2 (e^{-a_r t} - 1)^2 / (2a_r^2) + \int_0^t \sigma_r e^{-a_r (t-u)} d\widetilde{W}_r(u)\,, \tag{C.1}$$

Then we can write

$$\begin{aligned}\int_0^t r(u)du &= \int_0^t f(0,u)du + \int_0^t \sigma_r^2 (e^{-a_r u} - 1)^2 / (2a_r^2)du + \int_0^t \int_0^v \sigma_r e^{-a_r (v-u)} d\widetilde{W}_r(u)dv \\ &= \int_0^t f(0,u)du + \int_0^t \sigma_r^2 (e^{-a_r u} - 1)^2 / (2a_r^2)du + \int_0^t \int_u^t \sigma_r e^{-a_r (v-u)} dv d\widetilde{W}_r(u) \\ &= \int_0^t f(0,u)du + \int_0^t \sigma_r^2 (e^{-a_r u} - 1)^2 / (2a_r^2)du + \int_0^t (\sigma_r / a_r)\left[1 - e^{-a_r (t-u)}\right] d\widetilde{W}_r(u)\,.\end{aligned} \tag{C.2}$$

In a similar way, we restate equation (27) as follows:

$$s(t) = s(0,t) + \sigma_s^2 (e^{-a_s t} - 1)^2 / (2a_s^2) + \int_0^t \sigma_s e^{-a_s (t-u)} d\widetilde{W}_s(u)\,, \tag{C.3}$$

Hence we obtain

$$\int_0^t s(u)du = \int_0^t f(0,u)du + \int_0^t \sigma_s^2 (e^{-a_s u} - 1)^2 / (2a_s^2)du + \int_0^t (\sigma_s / a_s)\left[1 - e^{-a_s (t-u)}\right] d\widetilde{W}_s(u)\,. \tag{C.4}$$

The equations (C.1) to (C.4) are four normal random variables with the following means, variances, and covariances, repectively.

$$\mu_1(t) \equiv E\left(\int_0^t r(u)du\right) = \int_0^t f(0,u)du + \int_0^t \left[\sigma_r^2 \left(\exp\left(-a_r u\right) - 1\right)^2 / (2a_r^2)\right] du$$

$$\mu_2(t) \equiv E\left(r(t)\right) = f(0,t) + \sigma_r^2 \left(\exp\left(-a_r t\right) - 1\right)^2 / (2a_r^2)$$

$$\mu_3(t) \equiv E\left(\int_0^t s(u)du\right) = \int_0^t s(0,u)du + \int_0^t \left[\sigma_s^2 \left(\exp\left(-a_s u\right) - 1\right)^2 / (2a_s^2)\right] du$$

$$\mu_4(t) \equiv E\left(s(t)\right) = s(0,t) + \sigma_s^2 \left(\exp\left(-a_s t\right) - 1\right)^2 / (2a_s^2)$$

$$\begin{aligned}\sigma_1^2(t) &\equiv Var\left(\int_0^t r(u)du\right) \\ &= Var\left(\int_0^t (\sigma_r / a_r)\left[1 - \exp\left(-a_r (t-u)\right)\right] d\widetilde{W}_r(u)\right) \\ &= \int_0^t \left[\sigma_r^2 \left(1 - \exp\left(-a_r (t-u)\right)\right)^2 / a_r^2\right] du\end{aligned}$$

$$\sigma_2^2(t) \equiv Var\left(r(t)\right)$$

$$= Var\left(\int_0^t \sigma_r \left(-a_r(t-u)\right) d\widetilde{W_r}(u)\right)$$

$$= \int_0^t \sigma_r^2 \exp\left(-2a_r(t-u)\right) du$$

$$\sigma_3^2(t) \equiv Var\left(\int_0^t s(u)du\right)$$

$$= Var\left(\int_0^t (\sigma_s / a_s)\left[1-\exp\left(-a_s(t-u)\right)\right] d\widetilde{W_s}(u)\right)$$

$$= \int_0^t \left[\sigma_s^2 \left(1-\exp\left(-a_s(t-u)\right)\right)^2 / a_s^2\right] du$$

$$\sigma_4^2(t) \equiv Var\left(s(t)\right)$$

$$= Var\left(\int_0^t \sigma_s \left(-a_s(t-u)\right) d\widetilde{W_s}(u)\right)$$

$$= \int_0^t \sigma_s^2 \exp\left(-2a_s(t-u)\right) du$$

$$\sigma_{12}(t) \equiv \mathrm{cov}\left(\int_0^t r(u)du,\ r(t)\right)$$

$$= \mathrm{cov}\left(\int_0^t (\sigma_r / a_r)\left[1-\exp\left(-a_r(t-u)\right)\right] d\widetilde{W_r}(u),\ \int_0^t \sigma_r \exp\left(-a_r(t-u)\right) d\widetilde{W_r}(u)\right)$$

$$= \int_0^t (\sigma_r^2 / a_r) \exp\left(-a_r(t-u)\right)\left[1-\exp\left(-a_r(t-u)\right)\right] du$$

$$= \left(\sigma_r^2 / (2a_r^2)\right)\left(1-\exp\left(-a_r t\right)^2\right)$$

$$\sigma_{13}(t) \equiv \mathrm{cov}\left(\int_0^t r(u)du,\ \int_0^t s(u)du\right)$$

$$= \mathrm{cov}\left(\int_0^t \frac{\sigma_r}{a_r}\left[1-\exp\left(-a_r(t-u)\right)\right] d\widetilde{W_r}(u),\ \int_0^t \frac{\sigma_s}{a_s}\left[1-\exp\left(-a_s(t-u)\right)\right] d\widetilde{W_s}(u)\right)$$

$$= \int_0^t (\sigma_r \sigma_s \rho)\left[1-\exp\left(-a_r(t-u)\right)\right]\left[1-\exp\left(-a_s(t-u)\right)\right] / (a_r a_s) du$$

$$\sigma_{14}(t) \equiv \mathrm{cov}\left(\int_0^t r(u)du,\ s(t)\right)$$

$$= \mathrm{cov}\left(\int_0^t (\sigma_r / a_r)\left[1-\exp\left(-a_r(t-u)\right)\right] d\widetilde{W_r}(u),\ \int_0^t \sigma_s \exp\left(-a_s(t-u)\right) d\widetilde{W_s}(u)\right)$$

$$= \int_0^t (\sigma_r \sigma_s \rho)\left[1-\exp\left(-a_r(t-u)\right)\right] \exp\left(-a_s(t-u)\right) / a_r du$$

$$\sigma_{23}(t) \equiv \operatorname{cov}\left(r(t),\ \int_0^t s(u)du\right)$$

$$= \operatorname{cov}\left(\int_0^t \sigma_r \exp\left(-a_r(t-u)\right) d\widetilde{W}_r(u),\ \int_0^t (\sigma_s / a_s)\left[1-\exp\left(-a_s(t-u)\right)\right] d\widetilde{W}_s(u)\right)$$

$$= \int_0^t (\sigma_r \sigma_s \rho)\left[1-\exp\left(-a_s(t-u)\right)\right]\exp\left(-a_r(t-u)\right) / a_s du$$

$$\sigma_{24}(t) \equiv \operatorname{cov}\left(r(t), s(t)\right)$$

$$= \operatorname{cov}\left(\int_0^t \sigma_r \exp\left(-a_r(t-u)\right) d\widetilde{W}_r(u),\ \int_0^t \sigma_s \exp\left(-a_s(t-u)\right) d\widetilde{W}_s(u)\right)$$

$$= \int_0^t (\sigma_r \sigma_s \rho)\exp\left(-a_r(t-u) - a_s(t-u)\right) du$$

$$\sigma_{34}(t) \equiv \operatorname{cov}\left(\int_0^t s(u)du,\ s(t)\right)$$

$$= \operatorname{cov}\left(\int_0^t (\sigma_s / a_s)\left[1-\exp\left(-a_s(t-u)\right)\right] d\widetilde{W}_s(u),\ \int_0^t \sigma_s \exp\left(-a_s(t-u)\right) d\widetilde{W}_s(u)\right)$$

$$= \int_0^t (\sigma_s^2 / a_s)\exp\left(-a_s(t-u)\right)\left[1-\exp\left(-a_s(t-u)\right)\right] du$$

$$= \left(\sigma_s^2 /(2a_s^2)\right)\left(1-\exp\left(-a_s t\right)^2\right)$$