Genetics and/of basket options

Wolfgang Karl Härdle
Elena Silyakova

Ladislaus von Bortkiewicz
Chair of Statistics
C.A.S.E. Centre for Applied Statistics
and Economics
School of Business and Economics
Humboldt-Universität zu Berlin
http://lvb.wiwi.hu-berlin.de

Basket derivatives

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A basket of N assets with value at time t: $B(t) = \sum_{i=1}^{N} a_i S_i(t)$
Basket call: $\{B(T) - K_B\}^+$
where $S_i(t)$ - price of the i-th basket constituent at time t, a_i - quantity of the i-th asset, K_B - exercise price (strike) of a basket option, T - time of the option's expiry

Rainbow - best-of-N

$$\left[\max_{1\leq i\leq N}\{S_i(T)\} - K_B\right]^+$$

Mountain range - Atlas

$$\left\{\frac{1}{N-(N_1+N_2)}\sum_{j=1+N_1}^{N-N_2}\frac{S_j(T)}{S_j(0)}-K_B\right\}^+$$

where , N_1,N_2 - number of best and worst performing stocks.

Basket derivatives are popular

Cheaper than the corresponding portfolio of plain vanillas:

- portfolio effect
- smaller transaction costs

However create large correlation exposures.
Need to hedge!

Research questions

1. Which pricing model is suitable for multiasset options?
2. How to estimate dependence between basket assets?
3. How to estimate correlations in large dimensional baskets?

Outline

1. Motivation ✓
2. Basket dynamics in the Black-Scholes framework
3. Estimating correlation matrix
   - Historical (time series) correlation
   - Implied correlation
4. From equicorrelation to block correlation
5. Improving correlation estimates
6. Conclusion

Price dynamics of basket constituents

The price dynamic of the i-th stock in a basket is given by:

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i)dt + \sigma_i dW_i(t) \tag{1}$$

$$\rho_{ij} dt = dW_i(t) dW_j(t) \tag{2}$$

where r - interest rate, q_i - dividend yield of a stock i, σ_i - constant volatility of the i-th stock, ρ_{ij} - constant correlation between the i-th and the j-th stock, W - Brownian motion.

Dynamics of the basket's value

The dynamics of the basket's value is then given by:

$$\frac{dB(t)}{B(t)} = (r - q_B)dt + \frac{\sum_{i=1}^{N} w_i S_i(t)\sigma_i dW_i(t)}{\sum_{i=1}^{N} w_i S_i(t)} = \tag{3}$$
$$= (r - q_B)dt + dZ(t)$$

where q_B is the dividend yield of the basket and the relative weight w_i of the i-th constituent varies over time and is given by:

$$w_i = \frac{a_i S_i(t)}{\sum_{j=1}^{N} a_j S_j(t)} \tag{4}$$

Dynamics of correlated basket constituents

Correlation matrix

$$\Sigma = \begin{pmatrix} \rho_{11} & \cdots & \rho_{1N} \\ \vdots & \ddots & \vdots \\ \rho_{N1} & \cdots & \rho_{NN} \end{pmatrix}$$

Cholesky decomposition $\Sigma = MM^\top$ we obtain $M = (m_{i,j})_{1 \le i,j \le N}$, a lower triangular matrix, a "square root" of Σ.
The process for every individual asset S_i then:

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i)dt + \sigma_i \sum_{j=1}^{N} m_{i,j} dW_j(t) \tag{5}$$

Finally applying Itô's lemma we obtain the closed-form expression for simulation of the i-th stock process on a time interval $\Delta t = [t_1, t_2]$:

$$S_i(t_2) = S_i(t_1) \exp\left\{ (r - q_i - \frac{\sigma_i^2}{2})\Delta t + \sigma_i \sum_{j=1}^{N} m_{i,j} \sqrt{\Delta t} Z_j \right\} \quad (6)$$

where $Z_j \sim \mathrm{N}(0, 1)$, i.i.d.

Historical correlation

$X_i(t) = \log S_i(t) - \log S_i(t-1)$, log returns:

$$\rho_{ij} = \frac{\sum_{k=0}^{T} \lambda^k \{X_i(t-k) - \bar{X}_i(t)\}\{X_j(t-k) - \bar{X}_j(t)\}}{\sqrt{\sum_{k=0}^{T} \lambda^k \{X_i(t-k) - \bar{X}_i(t)\}^2 \sum_{k=0}^{T} \lambda^k \{X_j(t-k) - \bar{X}_j(t)\}^2}}$$

to obtain the historical correlation matrix

$$\begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{12} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1N} & \rho_{N2} & \cdots & 1 \end{bmatrix}$$

Here $\bar{X}_i(t)$ the arithmetic mean of the i-th log return calculated at time t, λ - decay parameter (RiskMetrics: $\lambda = 0.94$).

Equicorrelation matrix

$$\sigma^2_{Basket} = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}$$

replace $\begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \cdots & 1 \end{bmatrix}$ with $\begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}$,

then $\rho = \dfrac{\sigma^2_{Basket} - \sum_{i=1}^{N} w_i^2 \sigma_i^2}{2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} w_i w_j \sigma_i \sigma_j}$ is the basket correlation.

Nice property: for $-\{1/(N-1)\} < \rho < 1$ - positive definite (see Härdle, Simar, 2007).

Implied volatility

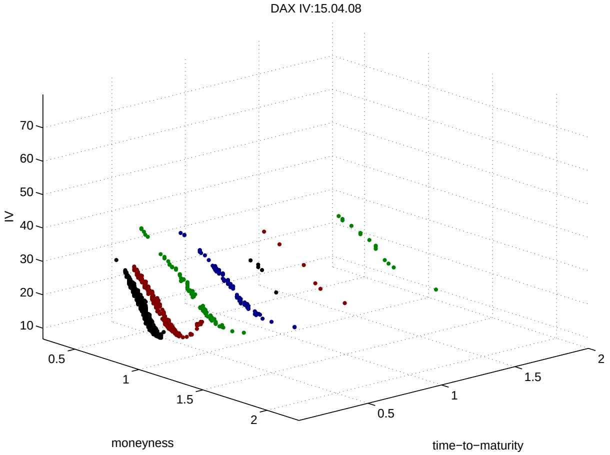
- consider an option with Black-Scholes price $V_{BS}\{S(t), \kappa, \sigma, \tau\}$
- given the observed market price $\widetilde{V}(t)$, the implied volatiltiy $\widehat{\sigma}$ can be found by solving:

$$V_{BS}\{S(t), \kappa, \widehat{\sigma}, \tau\} - \widetilde{V}(t, \kappa, \tau) = 0$$

,

where $\kappa = \frac{K}{S(t)\exp\{(r-d)\tau\}}$ - moneyness metric with K - strike of the option, τ - time to maturity of the option.

DAX implied volatility



Implied correlation

Using (8) map the implied volatility surfaces of a basket $\widehat{\sigma}_{Basket}(\kappa,\tau)$ and N constituents $\widehat{\sigma}_i(\kappa,\tau)$ to $\widehat{\rho}(\tau,\kappa)$ the **average implied correlation surface of a basket**:

$$\widehat{\rho}(\kappa,\tau) = \frac{\widehat{\sigma}^2_{Basket}(\kappa,\tau) - \sum_{i=1}^{N} w_i^2 \widehat{\sigma}_i^2(\kappa,\tau)}{2\sum_{i=1}^{N}\sum_{j=i+1}^{N} w_i w_j \widehat{\sigma}_i(\kappa,\tau)\widehat{\sigma}_j(\kappa,\tau)} \tag{7}$$

Dynamic modeling of correlation surfaces

Every t we observe $(X_{t,j}, Y_{t,j})$, $1 \leq j \leq J_t$, $1 \leq t \leq T$ where

- $Y_{t,j}$ - implied correlation
- $X_{t,j}$ - two-dimensional vector of κ and τ
- T - number of observed time periods (days)
- J_t - number of observations at day t

Dynamic modeling of correlation surfaces

In an orthogonal L-factor model, a J-dimensional random vector $Y_t = (Y_{t,1}, ..., Y_{t,J})$ can be represented as

$$Y_{t,j} = Z_{t,1} m_{1,j} + ... + Z_{t,L} m_{L,j} + \varepsilon_{t,j} \tag{8}$$

where
$Z_{t,j}$- common factors, $\varepsilon_{t,j}$ - errors and the coefficents $m_{l,j}$ - factor loadings.
It such setting the modelling of Y_t can be simplified to modelling of $Z_t = (Z_{t,1}, ..., Z_{t,L})$, which is more feasible for $L << J$.

Dynamic modeling of correlation surfaces

Including explanatory variables $X_{t,j}$ influencing the factor loadings $m_{l,j}$ rewrite (10)

$$Y_{t,j} = \sum_{l=1}^{L} Z_{t,l} m_l(X_{t,j}) + \varepsilon_{t,j} = Z_t^\top m(X_{t,j}) + \varepsilon_{t,j} \tag{9}$$

where

- $Z_t = (Z_{t1}, \ldots, Z_{tL})^\top$ - unobservable L-dimensional process
- m - L-tuple $(m_1, ..., m_L)$ of unknown real-valued functions
- $X_{t,j}, \ldots, X_{T,J_T}$ and $\varepsilon_{t,j}, \ldots, \varepsilon_{T,J_T}$ are independent
- $\varepsilon_{t,j}$ are $i.i.d.$ with zero mean and finite second moment

Dynamic modeling of correlation surfaces

$$Y_{t,j} = \sum_{l=1}^{L} Z_{t,l} \sum_{k=1}^{K} a_{l,k} \psi_k(X_{t,j}) + \varepsilon_{t,j} = Z_t^\top A \Psi_t + \varepsilon_t \qquad (10)$$

where

- A - $L \times K$ coefficient matrix
- $\Psi_t = \{\psi_1(X_t), ..., \psi_R(X_t)\}^\top$ - space basis, in Park et al. (2009) a tensor product of one dimensional B-spline basis.

Estimation

Define the least squares estimators $\widehat{Z}_t$ and $\widehat{A}$:

$$\sum_{t=1}^{T}\sum_{j=1}^{J}\left\{Y_{t,j} - Z_t^\top A\psi(X_{t,j})\right\}^2 = \min_{A,Z}!$$

Need to estimate space basis Ψ_t!

Choice of space basis

Estimate basis functions in a FPCA framework, motivated by Hall et. al (2006):
Find eigenfunctions corresponding to the K largest eigenvalues of the smoothed operator

$$\widehat{\psi}(u, v) = \widehat{\phi}(u, v) - \widehat{\mu}(u)\widehat{\mu}(v)$$

Choice of space basis

1. estimate $\widehat{\mu}(u)(\mu(v))$:

$$\sum_{t=1}^{T}\sum_{j=1}^{J}\{Y_{tj} - a - \sum_{c=1}^{2} b^c(u^c - X_{tj}^c)\}^2 K\left(\frac{X_{tj} - u}{h_\mu}\right)$$

2. estimate $\widehat{\phi}(u, v)$:

$$\sum_{t=1}^{T}\sum_{1\leq j\neq k\leq J_t}\{Y_{tj}Y_{tk} - a_0 - \sum_{c=1}^{2} b_1^c(u^c - X_{tj}^c) - \sum_{c=1}^{2} b_2^c(v^c - X_{tk}^c)\}^2$$

$$\times K\left(\frac{X_{tj} - u}{h_\phi}\right) K\left(\frac{X_{tj} - v}{h_\phi}\right)$$

3. compute $\widehat{\psi}(u, v) = \widehat{\phi}(u, v) - \widehat{\mu}(u)\widehat{\mu}(v)$ and take K eigenfunctions corresponding to the largest eigenvalues

Basis functions

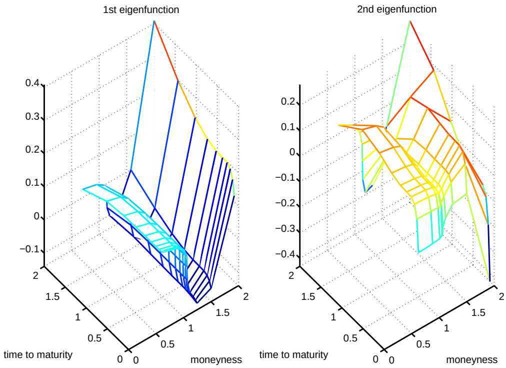


Figure 1: Eigenfunctions as basis functions estimated on 10x10 grid

Basis functions

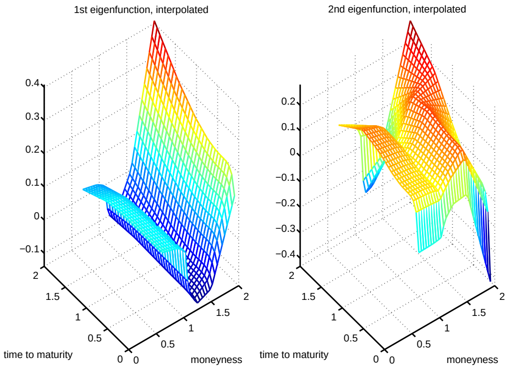
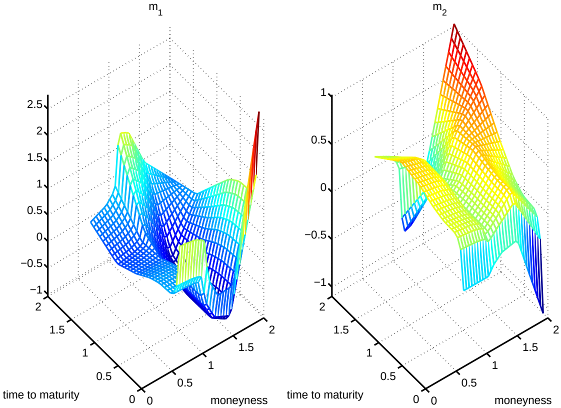
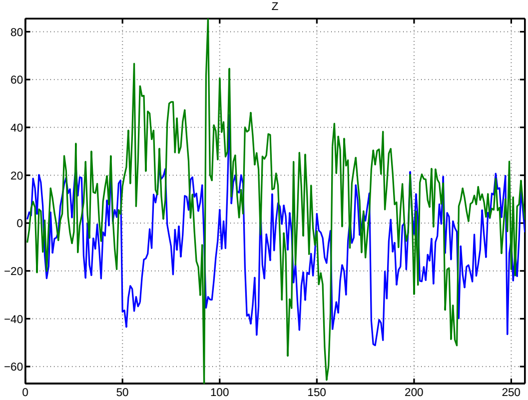


Figure 2: Eigenfunctions as basis functions interpolated

Estimated functions $\widehat{m}_l$



Estimated time series of factors $\widehat{Z}_{t1}$, $\widehat{Z}_{t2}$



From equicorrelation to block correlation

Group assets in the basket into k blocks, then

$$
\begin{bmatrix}
\begin{bmatrix}
1 & \rho_1 & \cdots & \rho_1 \\
\rho_1 & 1 & \cdots & \rho_1 \\
\vdots & \vdots & \ddots & \vdots \\
\rho_1 & \rho_1 & \cdots & 1
\end{bmatrix} & \cdots & \rho_{k+1} \\
\vdots & \ddots & \vdots \\
\rho_{k+1} & \cdots &
\begin{bmatrix}
1 & \rho_k & \cdots & \rho_k \\
\rho_k & 1 & \cdots & \rho_k \\
\vdots & \vdots & \ddots & \vdots \\
\rho_k & \rho_k & \cdots & 1
\end{bmatrix}
\end{bmatrix}
$$

Correlation matrix for 2 groups of assets (3 blocks)

$$\begin{bmatrix} \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_1 \\ \rho_1 & 1 & \cdots & \rho_1 \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1 & \rho_1 & \cdots & 1 \end{bmatrix} & \rho_3 \\ \rho_3 & \begin{bmatrix} 1 & \rho_2 & \cdots & \rho_2 \\ \rho_2 & 1 & \cdots & \rho_2 \\ \vdots & \vdots & \ddots & \vdots \\ \rho_2 & \rho_2 & \cdots & 1 \end{bmatrix} \end{bmatrix}$$

Block implied correlation, 3 blocks

$$\sigma^2_{Basket}(K,\tau) = \sum_{i=1}^{N} w_i^2 \sigma_i^2(K,\tau) +$$

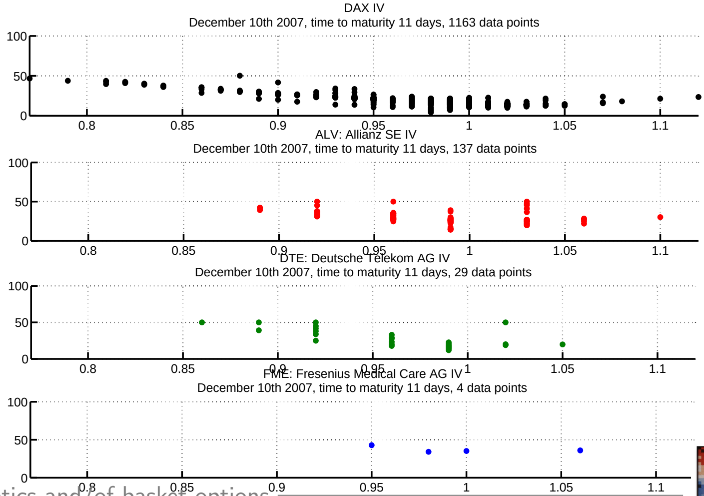
$$+2\sum_{i=1}^{M}\sum_{j=i+1}^{M} w_i w_j \sigma_i(K,\tau)\sigma_j(K,\tau)\rho_1(K,\tau) +$$

$$+2\sum_{i=1}^{N-M}\sum_{j=i+1}^{N-M} w_i w_j \sigma_i(K,\tau)\sigma_j(K,\tau)\rho_2(K,\tau) +$$

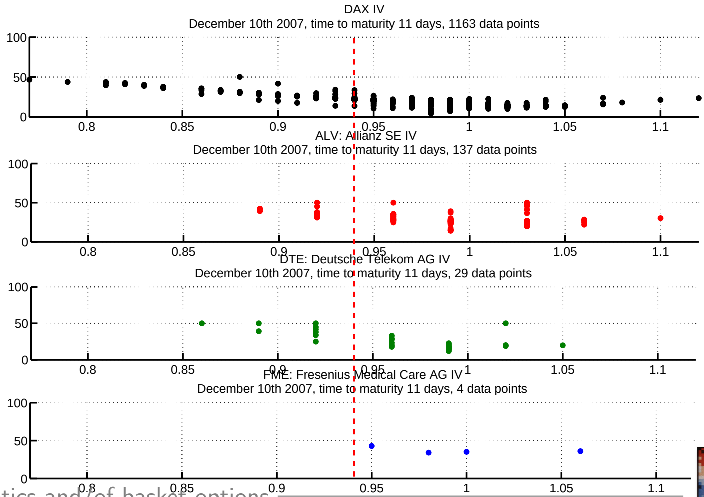
$$+2\sum_{i=1}^{M}\sum_{j=M+1}^{N} w_i w_j \sigma_i(K,\tau)\sigma_j(K,\tau)\rho_3(K,\tau)$$

where M - number of assets in the 1-st block.

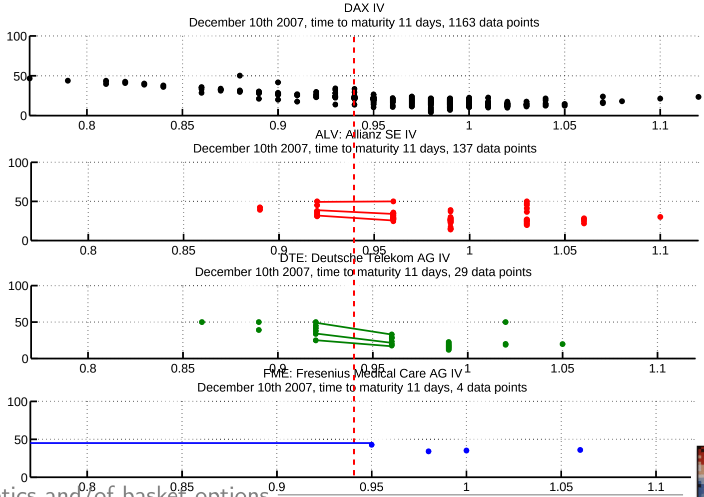
Synchronization in moneyness dimension



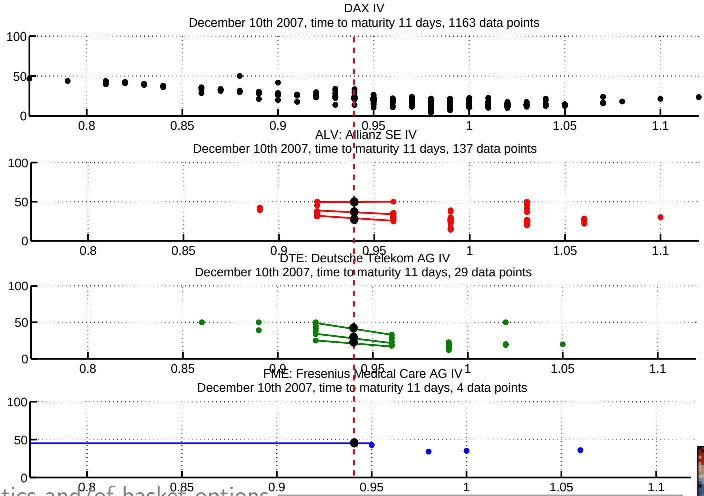
Synchronization in moneyness dimension



Synchronization in moneyness dimension



Synchronization in moneyness dimension



Challenges

- Moving to high-dimensional portfolios ($N \nearrow$) with block structure of covariance matrix:
  - need well-conditioned estimate of covariance matrix (Ledoit and Wolf (2003), Bickel and Levina (2008))
  - need to define the grouping procedure and way of finding the optimal block size (Hautsch, Kyj and Oomen (2009))
- Improving correlation surface modeling:
  - need to expand the time effect in a series model Z_t as a sum of basis functions (Song Härdle and Ritov (2010))

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