Pricing Defaultable bonds and CDS with PDE methods

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Chapter 1

Abstract

The market for credit derivatives is growing rapidly. The credit derivative market's global size was estimated to be $100 billion to $200 billion in 1996. The British Bankers Association estimated that the size was $1.6 trillion in 2001. Now the size is about $62 trillion [10]. The demand is strong because credit derivatives provide varieties that can fit different clients.

The fundamental credit derivative is the defaultable bond. When pricing defaultable bonds, we need to consider not only the face value and the coupon of the bonds but also the default risk. Bonds with high default risk should be cheaper than bonds with low default risk.

In this paper, we derive PDE models to price credit derivatives. In Chapter 2, we will discuss the reduced form approach, which is the popular method used in pricing a credit derivative. The reduced form approach models the default probability without considering the value of the firm [1]. The main idea is the yield spread, i.e. the difference between the yield of a defaultable bond and the yield of a default-free bond. Under the approach Li [13] concludes that the yield spread is attributed to two components, the default probability and the recovery rate, i.e. the fraction of the bond's face value paid in case of default. The reduced form approach is easy to calibrate because we can obtain the data of the yield spread from the market easily. However the yield spread is not only caused by these two components in the real world. Longstaff [15] mentions that the yield spread also may be caused by the liquidity.

In Chapter 3 we introduce the structural approach to price of defaultable bonds. The structural approach models default risk by modeling the value of a firm directly [1]. Under the structural approach, the pricing PDE is a 1-D PDE with a moving boundary. We assume that the interest rate is a constant in Chapter 3. But interest rates are an important factor when pricing bonds, so we introduce stochastic interest rate models in Chapter 4.

In Chapter 5, we review the papers that price defaultable bonds with a stochastic interest rate. One derives the fundamental PDE. There are analytic solutions for the PDEs, if the model of the interest rate is simple enough. If the model of the interest rate is complicated, however one has to use numerical methods in solving the PDEs.

We introduce a new method to price the most popular credit derivative in the market: the credit default swap (CDS) in Chapter 6. The CDS is a kind of insurance that protects the buyer of the CDS when a default event occurs. As a traditional insurance, the protection buyer makes regular premium payments quarterly or semiannually. When the default event occurs, the protection seller pays par value of the bond to the

buyer, the buyer physically delivers the bond to the seller, and the buyer ceases paying premiums. From the fundamental PDE in Chapter 5, we derive the PDE in Chapter 6 with a better interest rate model to price a CDS and discuss the issues we will face when solving it numerically.

Contents

Chapter 2

Pricing defaultable derivatives with the reduced form approach

In order to price defaultable bonds with the reduced form approach. Here we follow Li's paper [13] to construct the credit curve, which is the default probability of an firm over various time horizons. He uses real data from the market to plot the yield spread curve. From that, the credit curve can be constructed. Then he can price default derivatives.

The following are the steps to price the defaultable bonds by the reduced form approach: First, one defines the survival function. Assuming the default event occurs at time τ and $F(t) \equiv P(\tau \leq t)$ denotes the probability of the default event occurring before time t and

$$f(t) = F'(t), \tag{2.1}$$

is the default density function. Then we can define the {survival function} as

$$S(t) = 1 - F(t). \tag{2.2}$$

Here $S(t)$ means the probability that there is no default event in $[0, t]$. Then we know that the instantaneous marginal default probability is

$$P(x < \tau < x + \triangle x | \tau > x) = \frac{P(x < \tau < x + \triangle x)}{P(\tau > x)} \tag{2.3}$$

$$= \frac{F(x + \triangle x) - F(x)}{1 - F(x)} = \frac{f(x)}{1 - F(x)} \triangle x. \tag{2.4}$$

Here, we can define {default intensity function},

$$h(x) = \frac{f(x)}{1 - F(x)} = -\frac{S'(x)}{S(x)}. \tag{2.5}$$

We can rewrite $S(t)$, $F(t)$ and $f(t)$ as functions of the default intensity function. When we integrate both sides of (2.5),

$$S(t) = e^{-\int_0^t h(s)ds} \tag{2.6}$$

and

$$F(t) = 1 - S(t) = 1 - e^{-\int_0^t h(s)ds}. \tag{2.7}$$

Taking the derivative of (2.7),

$$f(t) = h(t) e^{-\int_0^t h(s)ds}. \tag{2.8}$$

Given that there is no default event in $[0, x]$, the probability that no default event occurs in $[x, t]$ is $P(\tau - x > t | \tau > x)$, which is denoted by ${}_t p_x$. We can see that ${}_t p_x$ is also a function of the default intensity function

$${}_t p_x = \frac{S(x+t)}{S(x)} = e^{-\int_x^{x+t} h(s)ds}. \tag{2.9}$$

If $t = 1$ we have notation that

$${}_1 p_x = p_x. \tag{2.10}$$

In order to price a defaultable bond, Li [13] assumes it pays coupons $C_1, C_2, ...C_n$ at times $t_1, t_2, ..., t_n$, and $D(t_0, t_1)$ is the discount factor. Let $R(t_i)$ be the recovery rate which means the fraction of the bond's face value paid in case of default. In the interval $[t_i, t_{i+1}]$, if there is no default event, the bond price is $C_{i+1} + V(t_{i+1})$, where $V(t_{i+1})$ is the bond price at time t_{i+1}. If the default event occurs in the $[t_i, t_{i+1}]$, the bond price is $R(t_i)[C_{i+1} + V(t_{i+1})]$. Assume that p_i is the marginal survival probability, Li have the bond price at t_i is

$$V(t_i) = \frac{D(t_0, t_{i+1})}{D(t_0, t_i)} p_i (C_{i+1} + V(t_{i+1})) + (1 - p_i) R(t_i) [C_{i+1} + V(t_{i+1})] \tag{2.11}$$

$$= \frac{D(t_0, t_{i+1})}{D(t_0, t_i)} \{[p_i + (1 - p_i) R(t_i)] [C_{i+1} + V(t_{i+1})]\}. \tag{2.12}$$

By recursion and $V(t_n) = 0$, we have

$$V(t_0) = \sum_{i=1}^{n} D(t_0, t_i) \left\{ \prod_{j=0}^{i-1} [p_j + (1 - p_j) R(t_{j+1})] \right\} C_i. \tag{2.13}$$

Let the credit discount factor be

$$DC(t_i) = \prod_{j=0}^{i-1} [p_j + (1 - p_j) R(t_{j+1})] \tag{2.14}$$

and the credit risk adjusted discount factor be

$$Q(t_0, t_1) = D(t_0, t_1) DC(t_i). \tag{2.15}$$

Here the credit discount factor is the fraction of coupon payments attributed to the probability of default. The credit risk adjusted discount factor is the discount factor that is adjusted by the credit discount factor. By rewriting (2.13), the price of a defaultable bond is

$$V(t_0) = \sum_{i=1}^{n} Q(t_0, t_1) C_i. \tag{2.16}$$

By using (2.9) and approximating $e^x \approx 1 + x$, Li [13] expands the discrete time model to the following continuous time model.

$$p_j + (1 - p_j) R(t_{j+1}) \approx e^{-(1 - R(t_{j+1})) \int_{t_j}^{t_{j+1}} h(s)ds}, \tag{2.17}$$

so

$$DC(t_i) = \prod_{j=0}^{i-1} [p_j + (1 - p_j) R(t_{j+1})] \approx e^{-\sum_{j=0}^{i-1} (1 - R(t_{j+1})) \int_{t_j}^{t_{j+1}} h(s)ds}. \tag{2.18}$$

If there are n partitions in (t_0, t_i),

$$\lim_{n \to \infty} DC(t_i) = e^{-\int_{t_0}^{t_i} [1 - R(t_{j+1})] h(s)ds}, \tag{2.19}$$

In the continuous time model, the discount factor is

$$D(t_0, t_i) = e^{-\int_{t_0}^{t_i} r(s)ds}. \tag{2.20}$$

Therefore the price of the defaultable bond is

$$V(t_0) = \sum_{i=1}^{n} C_i e^{-\int_{t_0}^{t_i} r(s)+(1-R(s))h(s)ds}. \tag{2.21}$$

Here $S_i = (1 - R(t_i))\, h(t_i)$ is called the yield spread or the risk premium. After having the risk premium, we can price default derivatives.

Reduced form models are characterized by strong data fitting ability, but they have poor predictive ability in the empirical study of Arora et al. [18]. Also, reduced form models underperform structural models across large and small firms in this empirical study. Furthermore, in reduced form models, the yield spread only has two components, namely the default intensity function and the recovery rate, but there are more than two components in the yield spread, like the liquidity mentioned in Longstaff's paper [15]. For these reasons, we are going to introduce structural models in the next chapter.

Chapter 3

Pricing a defaultable bond with structural approaches

In order to price a defaultable bond with structural approaches, one assumes that the value of the firm's assets follows a stochastic process. If the value of the firm is lower than a certain threshold, the firm would immediately default. There are two popular structural models. One is Merton's model [16]. Merton assumes that the value of the firm's assets is a stochastic process and the default event means the stochastic process touches the default boundary. He assumes that the value of the firm's assets follows a Geometric Brownian motion, so that the pricing formula of defaultable bonds is the same as the pricing formula of European options. The other is Black and Cox's model [2]. Whereas Merton assumes that a default event can only occur at the maturity date, Black and Cox assume that default may occur before that. For that reason, the pricing PDE of Black and Cox's model has a moving boundary, but Merton's model does not. We will discuss the details in following sections.

3.1 Black and Cox's model

Black and Cox [2] develop a model that allows a company to default before maturity. In this case, if we assume that the value process V is a Geometric Brownian motion, r is the constant risk-free rate, $u(V,t)$ is the price of the defaultable bond and L is the par value of the defaultable bond, there is a PDE that can be used in pricing the credit risk derivatives:

$$u_t(V,t) + (r-k)\,V u_V(V,t) + \frac{1}{2}\sigma_V^2 V^2 u_{VV}(V,t) - ru(V,t) = 0, \tag{3.1}$$

with the final condition

$$u(V,T) = min\ (V,L)\,, \tag{3.2}$$

and the boundary condition

$$u\left(Ke^{-r(T-t)},t\right) = \beta L e^{-r(T-t)} = Ce^{-r(T-t)}, \tag{3.3}$$

where β is the recovery rate and $\beta L = C$. Furthermore, k is the constant payout ratio (dividend). The parameter K is a quantity given in the safety covenant, where K satisfies $0 < K < L$. If K is large, the bond buyer is highly protected. The bond is easy to default, but the bond buyer can receive almost as much as L when the default occurs. The default condition is $V < v_t$, where $v_t = Ke^{-r(T-t)}$ for $t < T$ and $v_t = L$ for $t = T$. Here Black and Cox assume $K = C$.

3.2 Solving the moving boundary problem

From the last section, we see that Black and Cox's [2] model generates a moving boundary condition. However the moving boundary condition is an issue when approximating the solution numerically. As the boundary moves, there is a gap between the previous boundary and the boundary in next time step. We have to interpolate the mesh in this gap which causes error. For that reason, we try to avoid the moving boundary by fixing the boundary. In order to fix the boundary, we change variables from V to ε. ε here is always between 0 and 1, so the boundary does not move in the transformed space. Additionally, after changing variables, the coefficients become the function of τ, for

$$u\left(V,t\right)=u\left(\varepsilon,\tau\right) \tag{3.4}$$

and

$$A\left(t\right)=Ce^{-r(T-t)} \tag{3.5}$$

by

$$V=A\left(t\right)+\varepsilon\left(V_{max}-A\left(t\right)\right) \tag{3.6}$$

$$\tau=T-t \quad (Solve\ it\ backward). \tag{3.7}$$

By the chain rule we have

$$\frac{\partial u}{\partial V}=\frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial V}=\frac{\partial u}{\partial \varepsilon}\frac{1}{V_{max}-A\left(T-\tau\right)} \tag{3.8}$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t}+\frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} \tag{3.9}$$

$$=-\frac{\partial u}{\partial \tau}+\frac{\partial u}{\partial \varepsilon}\frac{A'\left(T-\tau\right)\left(V-V_{\max}\right)}{\left(V_{max}-A\left(T-\tau\right)\right)^{2}} \tag{3.10}$$

$$\frac{\partial^2 u}{\partial V^2}=\frac{\partial}{\partial V}\left(\frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial V}\right)=\frac{\partial^2 u}{\partial \varepsilon^2}\left(\frac{1}{V_{max}-A\left(T-\tau\right)}\right)^2. \tag{3.11}$$

So the original PDE with a moving boundary becomes

$$-u_\tau+\left[\frac{\left(r-k\right)V}{V_{max}-A\left(T-\tau\right)}+\frac{\left(V-V_{max}\right)A'\left(T-\tau\right)}{\left(V_{max}-A\left(T-\tau\right)\right)^{2}}\right]u_\varepsilon+\frac{1}{2}\sigma^2\left[A\left(T-\tau\right)+\varepsilon\left(V_{max}-A\left(T-\tau\right)\right)\right]^2u_{\varepsilon\varepsilon}-ru=0 \tag{3.12}$$

with fixed boundary,

$$u(0,\tau)=A(T-\tau) \quad and \quad u(1,\tau)=e^{-r\tau}. \tag{3.13}$$

3.3 Numerical solutions

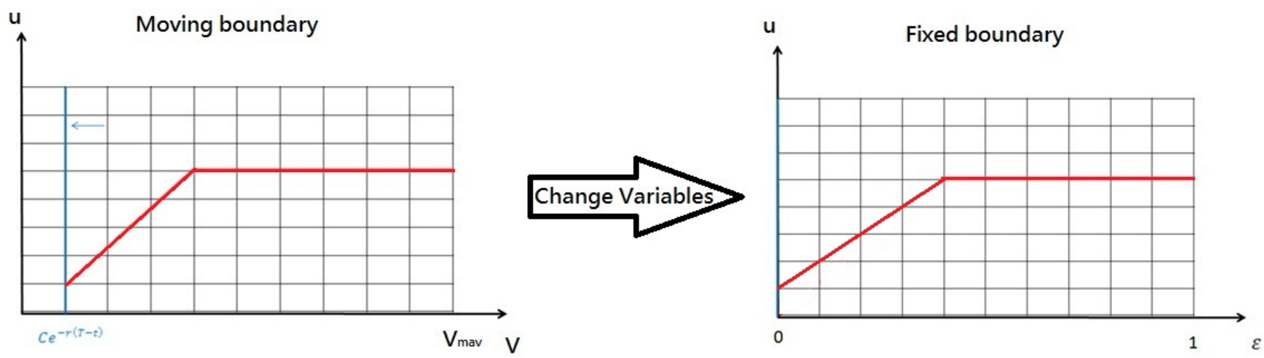
In this section, we show that the pricing PDE can be solved numerically by comparing the numerical solution and the analytic solution. Thus, we can use the same method to solve the problem that does not have analytic solutions in Chapter 5.

3.3.1 The method

In order to solve the PDE (3.12) numerically, we define

$$p\left(\varepsilon,\tau\right)=\frac{\left(r-k\right)V}{V_{max}-A\left(T-\tau\right)}+\frac{\left(V-V_{max}\right)A'\left(T-\tau\right)}{\left(V_{max}-A\left(T-\tau\right)\right)^{2}} \tag{3.14}$$

Figure 3.1: Changing variables from V to ε



$$q(\varepsilon,\tau) = \frac{1}{2}\sigma^2 \left[A(T-\tau) + \varepsilon\left(V_{max} - A(T-\tau)\right)\right]^2, \tag{3.15}$$

so that the PDE becomes

$$u_\tau = p(\varepsilon,\tau)\, u_\varepsilon + q(\varepsilon,\tau)\, u_{\varepsilon\varepsilon} - ru. \tag{3.16}$$

We can approximate the PDE (3.16) with finite difference schemes. For instance, to second order,

$$u_\varepsilon|\varepsilon_j = \frac{u_{j+1} - u_{j-1}}{2\Delta\varepsilon} + O\left(\Delta\varepsilon^2\right) \tag{3.17}$$

and

$$u_{\varepsilon\varepsilon}|\varepsilon_j = \delta_\varepsilon^+ \delta_\varepsilon^- u(\varepsilon_j) + O\left(\Delta\varepsilon^2\right). \tag{3.18}$$

Then

$$\frac{\partial u}{\partial \tau}|_j = a_j u_{j-1} + b_j u_j + c_j u_{j+1}, \tag{3.19}$$

where

$$a_j = \frac{p(\varepsilon_j,\tau_n)}{2\Delta\varepsilon} - \frac{q(\varepsilon_j,\tau_n)}{\Delta\varepsilon^2} \tag{3.20}$$

$$b_j = r + \frac{2q(\varepsilon_j,\tau_n)}{\Delta\varepsilon^2} \tag{3.21}$$

$$c_j = -\frac{p(\varepsilon_j,\tau_n)}{2\Delta\varepsilon} - \frac{q(\varepsilon_j,\tau_n)}{\Delta\varepsilon^2}. \tag{3.22}$$

Generic Dirichlet boundary conditions are

$$u_o(\tau) = g^L(\tau) \tag{3.23}$$

$$u_J(\tau) = g^R(\tau). \tag{3.24}$$

In order to integrate in time by the trapezoidal rule, we assume

$$F(\tau_n, u^n) = a_j u_{j-1}^n + b_j u_j^n + c_j u_{j+1}^n. \tag{3.25}$$

So

$$u^{n+1} = u^n + \frac{\Delta\tau}{2}\left(F\left(\tau_{n+1}, u^{n+1}\right) + F(\tau_n, u^n)\right). \tag{3.26}$$

Collecting u^{n+1} on left hand side, we have

$$-\frac{\Delta t}{2} a_j u_{j-1}^{n+1} + \left(1 - \frac{\Delta t}{2} b_j\right) u_j^{n+1} - \frac{\Delta t}{2} c_j^u u_{j+1}^{n+1} = \frac{\Delta t}{2} a_j^u u_{j-1}^n + \left(1 + \frac{\Delta t}{2} b_j\right) u_j^n + \frac{\Delta t}{2} c_j u_{j+1}^n. \tag{3.27}$$

Rewriting, we have

$$M_1\overrightarrow{u^{n+1}} = M_2\overrightarrow{u^n} + \Delta\tau\overrightarrow{g} = \overrightarrow{RHS_j^n} \tag{3.28}$$

and formally,

$$\overrightarrow{u^{n+1}} = M_1^{-1}\overrightarrow{RHS_j^n} \tag{3.29}$$

where

$$\overrightarrow{u^n} = (u_0^n, u_1^n, ..., u_J^n) \tag{3.30}$$

From (3.29), we can see that (3.28) problem requires a matrix system to be solved. Here we can use the following Algorithm 1:

1 *Set the initial values* $u_j^0; j = 0, 1, 2, \ldots, N_x$.
2 **for** $n = 0$ *to* N_T **do**
3 $\quad M_1\overrightarrow{u^{n+1}} = M_2\overrightarrow{u^n} + \Delta\tau\overrightarrow{g} = \overrightarrow{RHS_j^n}$
4 $\quad u^{n+1} = M_1^{-1}\overrightarrow{RHS_j^n}$.
5 **end**
6 $v = A(t) + \varepsilon(v_{\max} - A(t))$ *and* $t = \tau$

Algorithm 1: Alogrithm of Solving IBVP PDE

3.3.2 Order of accuracy

In order to prove the numerical scheme is accurate to second order, we find the local truncation error.

Definition 3.3.1. Local Truncation Error

If we approximate the PDE $Pu(t,x) = f$ with an approximation $P_hU = R_hf$, then the local truncation error, τ_e is

$$\tau_e = P_h(\phi) - R_h(P\phi) \tag{3.31}$$

for any smooth function $\phi(t,x)$.

In order to derive local truncation error, here we have

$$P_h = \delta_t^+ + p\left(\frac{I + S_t^+}{2}\right)\left(\frac{S_t^+ - S_t^+}{2\triangle\varepsilon}\right) + q\left(\frac{I + S_t^+}{2}\right)\delta_t^+\delta_t^- - r\left(\frac{I + S_t^+}{2}\right) \tag{3.32}$$

and

$$R_h = \left(\frac{I + S_t^+}{2}\right). \tag{3.33}$$

Expanding (3.32),

$$P_h = \left(\frac{\left(I + \triangle t D_t + \frac{\triangle t^2}{2} D_t^2 + ...\right) - I}{\triangle t} \right)$$

$$+p \left(\frac{I + \left(I + \triangle t D_t + \frac{\triangle t^2}{2} D_t^2 + ...\right)}{2} \right) \left(\frac{\left(I + \triangle x D_x + \frac{\triangle x^2}{2} D_x^2 + ...\right) - \left(I - \triangle x D_x + \frac{\triangle x^2}{2} D_x^2 - ...\right)}{2\triangle x} \right)$$

$$+q \left(\frac{I + \left(I + \triangle t D_t + \frac{\triangle t^2}{2} D_t^2 + ...\right)}{2} \right) \left(\frac{\left(I + \triangle x D_x + \frac{\triangle x^2}{2} D_x^2 + ...\right) - I}{\triangle x} \right) \left(\frac{I - \left(I - \triangle x D_x + \frac{\triangle x^2}{2} D_x^2 - ...\right)}{\triangle x} \right)$$

$$-r \left(\frac{I + \left(I + \triangle t D_t + \frac{\triangle t^2}{2} D_t^2 + ...\right)}{2} \right) \tag{3.34}$$

$$= \left(D_t + \frac{\triangle t}{2} D_t^2 + \frac{\triangle t^2}{6} D_t^3 + ... \right) + p \left(I + \frac{\triangle t}{2} D_t + \frac{\triangle t^2}{4} D_t^2 + ... \right) \left(D_x + \frac{\triangle x^2}{6} D_x^3 + ... \right)$$

$$+q \left(I + \frac{\triangle t}{2} D_t + \frac{\triangle t^2}{4} D_t^2 + ... \right) \left(D_x + \frac{\triangle x}{2} D_x^2 + \frac{\triangle x^2}{6} D_x^3 + ... \right) \left(D_x - \frac{\triangle x}{2} D_x^2 + \frac{\triangle x^2}{6} D_x^3 - ... \right)$$

$$-r \left(I + \frac{\triangle t}{2} D_t + \frac{\triangle t^2}{4} D_t^2 + ... \right) \tag{3.35}$$

Also, we can expand (3.33),

$$R_h(P\phi) = \left(\frac{I + \left(I + \triangle t D_t + \frac{\triangle t^2}{2} D_t^2 + ...\right)}{2} \right) \left(D_t + p D_x + q D_x^2 - rI \right) \phi \tag{3.36}$$

$$= \left(I + \frac{\triangle t}{2} D_t + \frac{\triangle t^2}{4} D_t^2 + ... \right) \left(D_t + p D_x + q D_x^2 - rI \right) \phi. \tag{3.37}$$

So

$$\tau_e = P_h(\phi) - R_h(P\phi) = \frac{1}{12} \left(\triangle t^2 D_t^3 \right) + p \left(\frac{\triangle x^2}{6} D_x^3 \right) + q \left(-\frac{\triangle x^2}{4} D_x^2 \right) + O\left(\triangle t^3\right) + O(\triangle x^3) \tag{3.38}$$

$$= O\left(\triangle t^2\right) + O(\triangle x^2). \tag{3.39}$$

If we let $\triangle t = O(\triangle x)$,

$$\tau_e = O(\triangle x^2). \tag{3.40}$$

Therefore, the numerical scheme is accurate to second order.

3.3.3 Comparing the numerical solution with the analytic solution

We can compare the numerical solution with the analytic solution from Black and Cox [2]. The defaultable corporate bond price is

$$u(V,t) = Le^{-r(T-t)}\left[N(z_1) - y^{2\theta-2}N(z_2)\right]$$
$$+Ve^{-a(T-t)}\left[N(z_3) + y^{2\theta}N(z_4) + y^{\theta+\xi}e^{a(T-t)}N(z_5) + y^{\theta-\xi}e^{a(T-t)}N(z_6) - y^{\theta+\xi}N(z_7) - y^{\theta+\xi}N(z_8)\right] \tag{3.41}$$

where

$$y = Ce^{r(T-t)}/V \tag{3.42}$$

$$\theta = \frac{-a + 0.5\sigma^2}{\sigma^2} \tag{3.43}$$

$$\delta = \left(a + 0.5\sigma^2\right)^2 \tag{3.44}$$

$$\xi = \frac{\sqrt{\delta}}{\sigma^2} \tag{3.45}$$

$$\eta = \frac{\sqrt{\delta - 2\sigma^2 a}}{\sigma^2} \tag{3.46}$$

$$z_1 = \frac{log(V) - log(L) + \left(r - a - 0.5\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}} \tag{3.47}$$

$$z_2 = \frac{log(V) - log(L) + 2log(y) + \left(r - a - 0.5\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}} \tag{3.48}$$

$$z_3 = \frac{-log(V) + log(L) + \left(r - a + 0.5\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}} \tag{3.49}$$

$$z_4 = \frac{log(V) - log(L) + 2log(y) + \left(r - a + 0.5\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}} \tag{3.50}$$

$$z_5 = \frac{log(y) + \xi\sigma^2(T-t)}{\sqrt{\sigma^2(T-t)}} \tag{3.51}$$

$$z_6 = \frac{log(y) - \xi\sigma^2(T-t)}{\sqrt{\sigma^2(T-t)}} \tag{3.52}$$

$$z_7 = \frac{log(y) + \eta\sigma^2(T-t)}{\sqrt{\sigma^2(T-t)}} \tag{3.53}$$

$$z_8 = \frac{log(y) - \eta\sigma^2(T-t)}{\sqrt{\sigma^2(T-t)}}. \tag{3.54}$$

Let

$$T = 0.5 \tag{3.55}$$

$$r = 0.05 \tag{3.56}$$

$$\sigma = 0.2 \tag{3.57}$$

$$k = 0.06 \tag{3.58}$$

$$L = 10 \tag{3.59}$$

$$C = 0.8 \tag{3.60}$$

$$V_{max} = 40 \tag{3.61}$$

In Tables 3.1 and 3.2, we compare the analytic solutions and the numerical solutions with 160 mesh points. In order to make sure the result converges with second order, we run the routine with mesh points, Nx= 160, 320, 640, and 1280 and show the results in Table 3.2. Here we define that the error is the maximum of error between the analytic and the numerical solutions. In Figure 3.3 We can see that the log(error) has slope about -2 which means that it converges with order 2.

In conclusion, we can price a defaultable bond analytically and numerically under Black and Cox's model. Thus, we can use the same method in solving PDE of more complex model that does not have analytic solution in Chapter 5.

Table 3.1: The analytic and numerical solutions of Black and Cox's PDE with 160 mesh points.

v	Analytic Solutions	Numerical Solutions
2	1.94089	1.94089
4	3.88178	3.88178
6	5.82264	5.82263
8	7.73589	7.73544
10	9.18000	9.18011
12	9.67760	9.67707
14	9.74787	9.74763
16	9.75287	9.75284
:	:	:
38	9.75310	9.75310
40	9.75310	9.75310
Max error		5.27E-04

Table 3.2: The max error versus the number of mesh points for the solution of Black and Cox's PDE.

Mesh points	Error of numerical solutions
160	5.2660E-04
320	1.5535E-04
640	5.6595E-05
1280	1.6595E-05

Figure 3.2: The analytic and the numerical solutions of Black and Cox's PDE with 160 mesh points.

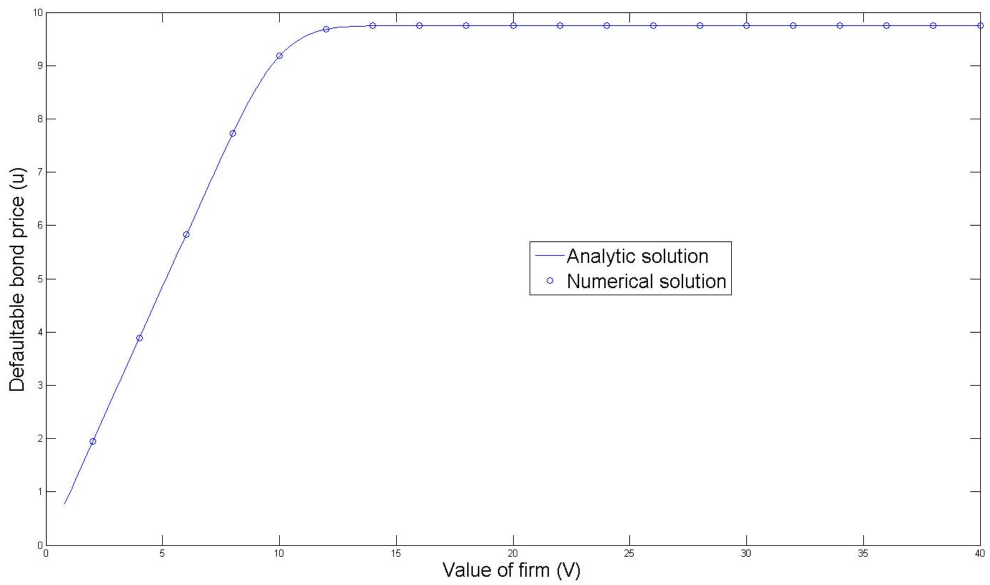
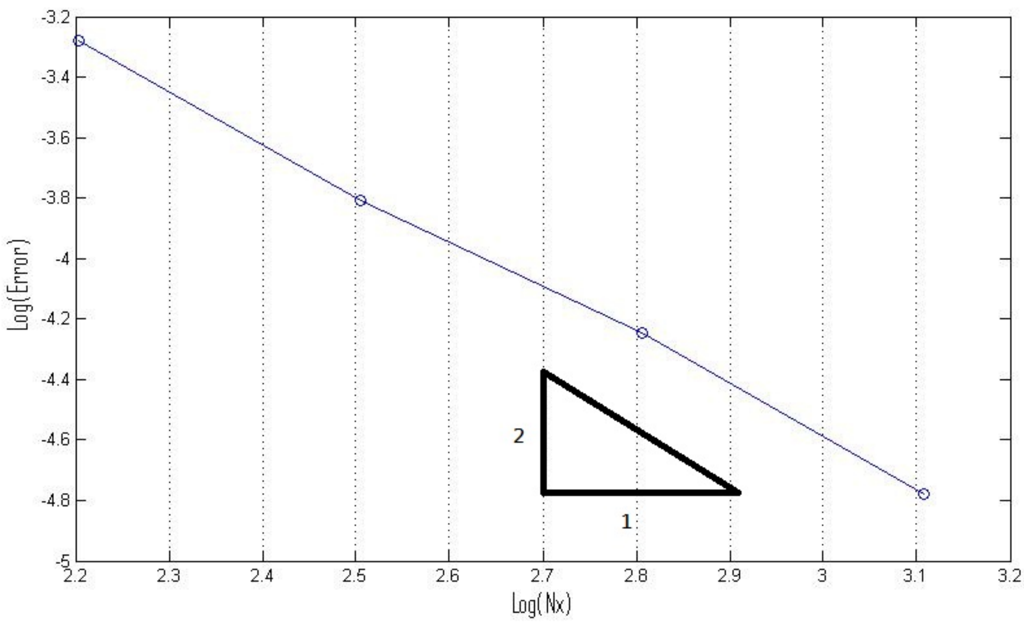


Figure 3.3: Logarithm of the max error versus logarithm of the number of mesh points for the solution of Black and Cox's PDE.



Chapter 4

Models of the interest rate

The interest rate is a constant under Black and Cox's model which may be not true when the length of time until the maturity date is long. In order to be closer to the real world, we would like to involve the stochastic interest rate. Thus, we discuss the popular stochastic interest rate models in this chapter. A good interest rate model should have following properties: First, it can be calibrated to describe the current term structure of interest rate. Second, it can fit the term structure of the volatilities. Four popular interest rate models are mentioned in Brigo's book [5]:

1. Vasicek's model [21]
2. Cox, Ingersoll, and Ross' model [8] (the CIR model)
3. Hull-White's model [11]
4. Black, Derman and Toy's model [3] (the BDT model)

We will review those models in this chapter, and we will find that not all of these four models have the desired properties.

4.1 The Vasicek model

In Vasicek's model [21], the interest rate is a stochastic process driven by only one factor, $\tilde{\omega}_t$. Here $\{\tilde{\omega}_t, t > 0\}$ is a standard Brownian motion.

Definition 4.1.1. Standard Brownian motion

1. $\tilde{\omega}_0 = 0$.
2. $\tilde{\omega}_t$ is continuous almost surely.
3. $\tilde{\omega}_t$ has independent increments with $\tilde{\omega}_t - \tilde{\omega}_s \sim N(0, t-s)$ (for $0 \leq$ s $<$ t).

Definition 4.1.2. Vasicek's model [21]
The short rate follows the process

$$dr_t = (\alpha - \beta r_t)\, dt + \eta d\tilde{\omega}_t \tag{4.1}$$

The short rate, r_t, used here is the (annualized) interest rate for an infinitesimally short period of time from time t. It is important, because after we know the model of the short rate, we can price a default-free zero coupon bond, $B(t,T)$, at t with maturity date T, as

$$B(t,T) = E\left\{\exp\left(-\int_t^T r_u du\right)\right\}. \tag{4.2}$$

The advantages:

- The model is famous for capturing the mean reversion of interest rate data, an essential characteristic of the interest rate that sets interest rate data apart from other financial prices.
- r_t's Probability Density Function (PDF) has closed form solution,

$$r_t = e^{-\alpha t}\left(r_0 + \beta\left(e^{\alpha t} - 1\right)\eta \int_0^t e^{\alpha n} d\omega_n^p\right) \quad (Ornstein - Uhlenbeck\, process)\,; \tag{4.3}$$

The disadvantages:

- r_t could be negative.
- It cannot capture the term structure well .

4.2 The Cox, Ingersoll and Ross (CIR) model

In the Cox-Ingersoll-Ross (CIR) model [8], the coefficient of $d\tilde{\omega}_t$ is not constant. That change is made to guarantee that r_t is positive, which is the main characteristic of the CIR model.

Definition 4.2.1. The CIR model [8]
The short rate follows

$$dr_t = (\alpha - \beta r_t)\,dt + \eta\sqrt{r_t}d\tilde{\omega}_t. \tag{4.4}$$

The advantages:

- The r_t has to be positive.
- r_t's PDF is a noncentral Chi-square distribution, i.e. r_t can be found in closed form.

The disadvantages:

- It cannot capture the term structure well.

4.3 The Hull and White model

Hull and White's model [11] is also called the extended Vasicek model. The constant coefficients in the Vasicek model become functions of t. In other words, α, β and η are time-dependent.

Definition 4.3.1. The Hull-White model [11]
The short rate follows

$$dr_t = (\alpha_t - \beta_t r_t)\, dt + \eta_t d\tilde{\omega}_t \tag{4.5}$$

The advantages:

- It can capture term structure well, because they extend the α, β and η to be functions of t.

The disadvantages:

- It does not exclude negative interest rates.

4.4 The Black, Derman and Toy (BDT) model

Black, Derman and Toy's model [3] is one of most popular interest rate models in the industry because it is easy in calibrating. Under the BDT model, by using binomial trees, it can capture both the term structure of the interest rate and the term structure of the volatility of the interest rate caps. So the BDT model is useful when pricing more complex interest-rate sensitive securities. Black and Karasinski [4] extend the BDT model to continuous-time.

Definition 4.4.1. The Black-Derman-Toy (BDT) model
The short rate follows

$$dln(r_t) = a_t\,(b_t - ln(r_t))\, dt + \eta_t d\tilde{\omega}_t. \tag{4.6}$$

The advantages:

- It can capture term structure well.
- r_t can not be negative.

The disadvantages:

- There is no analytic solution for the price of bonds.

After review the stochastic interest rate models, we can see the advantages and disadvantages of the models. So we can involve the stochastic interest rate when pricing credit derivatives in next Chapter.

Chapter 5

Pricing credit derivatives with a stochastic interest rate by PDE methods

In Chapter 4, we can see the advantages and disadvantages of the interest rate models. Furthermore, all models can be written in the form of $dr_t = \mu_r(r_t, t)\,dt + \sigma_r(r_t, t)\,d\widetilde{\omega}_t$. In this chapter we will derive the fundamental PDE with stochastic interest rate for all credit derivatives and review papers which involved the stochastic interest rate models.

5.1 Derivation of the fundamental PDE

In order to derive the PDE, Bielecki and Rutkowski [1] make the following assumptions:

1. The risk-neutral dynamics of the short-term interest rate processes $r_t, t > 0$, are given as

$$dr_t = \mu_r(r_t, t)\,dt + \sigma_r(r_t, t)\,d\widetilde{\omega}_t \tag{5.1}$$

   where $\widetilde{\omega}_t$ is standard Brownian motion defined in Def. 4.1.1.

2. By Assumption (5.1), Musiela and Rutkowski [17] find that a unit default-free zero coupon bond, B(t,T), should follow

$$dB(t,T) = B(t,T)\left[r_t dt + \sigma_B(r_t, t, T)\,d\widetilde{\omega}_t\right]. \tag{5.2}$$

3. V_t is the total value of the firm's asset at time t, and is given as

$$\frac{dV_t}{V_t} = \left[r_t - k(V_t, r_t, t)\right]dt + \sigma_V(V_t, t)\,d\omega_t^*, \tag{5.3}$$

   where k is the constant payout ratio (dividend).

4. The promised contingent claim, X, representing the firm's liabilities is to be redeemed at the maturity date T.

5. The recovery claim is $\widetilde{X}$, which represents the recovery payoff received at τ if default occurs prior to or at T.

6. The default triggering barrier process v equals

$$v_t = \bar{v}\left(V_t, r_t, t\right). \tag{5.4}$$

7. The default time is

$$\tau = inf\{t > 0; V_t < v_t\}. \tag{5.5}$$

8. The price process of the defaultable claim is

$$X^d\left(T\right) = X1_{\{\tau > T\}} + \widetilde{X}1_{\{\tau \leqslant T\}}. \tag{5.6}$$

9. $B\left(t, T\right)$ is the price of a unit default-free zero-coupon bond maturing at T.

10. The savings account S_t follows

$$S_t = \exp\left(\int_0^t r_u du\right). \tag{5.7}$$

In order to derive the fundamental PDE, one examines a self-financing trading strategy with a portfolio. Assume we have weights,

$$\phi_t = \left(\phi_t^0, \phi_t^1, \phi_t^2, \phi_t^3\right), \tag{5.8}$$

which generate the portfolio $U_t\left(\phi\right)$

$$U_t\left(\phi\right) = \phi_t^0 X^d\left(t, T\right) + \phi_t^1 V_t + \phi_t^2 B\left(t, T\right) + \phi_t^3 S_t. \tag{5.9}$$

A portfolio is self-financing if there is no money withdrawn from or deposited to it. Mathematically,

$$dU_t\left(\phi\right) = \phi_t^0\left[dX^d\left(t, T\right) + c\left(V_t, r_t, t\right)dt\right] + \phi_t^1\left[dV_t + k\left(V_t, r_t, t\right)V_t dt\right] + \phi_t^2 dB\left(t, T\right) + \phi_t^3 dS_t, \tag{5.10}$$

where $c\left(V_t, r_t, t\right)$ is the coupon rate and the firm is assumed to pay cash flows continuously at the rate $k\left(V_t, r_t, t\right)$.

In order to replicate X^d by V_t, $B\left(t, T\right)$ and S_t, $U_t\left(\phi\right)$ should be zero. After solving for ϕ_t^3,

$$\phi_t^3 = -S_t^{-1}\left[\phi_t^0 X^d\left(t, T\right) + \phi_t^1 V_t + \phi_t^2 B\left(t, T\right)\right]. \tag{5.11}$$

By substituting (5.14) for ϕ_t^3 into (5.10) and using $dS_t = r_t S_t dt$, we obtain

$$\phi_t^0[dX^d\left(t, T\right) + c\left(V_t, r_t, t\right)dt] + \phi_t^1[dV_t + k\left(V_t, r_t, t\right)V_t dt]$$

$$+\phi_t^2 dB\left(t, T\right) - r_t\left[\phi_t^0 X^d\left(t, T\right) + \phi_t^1 V_t + \phi_t^2 B\left(t, T\right)\right]dt = 0. \tag{5.12}$$

Bielecki and Rutkowski [1] prove that X^d is a function of V_t, r_t and t. So we let $X^d = u\left(V_t, r_t, t\right)$, then

$$dX^d\left(t, T\right) = du\left(V_t, r_t, t\right) \tag{5.13}$$

$$= \mu_x\left(t\right)dt + \sigma_{x,v}\left(t\right)d\omega_t^* + \sigma_{x,r}\left(t\right)d\widetilde{\omega}_t. \tag{5.14}$$

By Itô's lemma,

$$\mu_x\left(t\right) = u_V\left(r_t - k\left(V_t, r_t, t\right)\right)V_t + u_r\mu\left(r_t, t\right) + \frac{1}{2}u_{VV}\sigma_V^2\left(V_t, t\right)V_t^2$$

$$+\frac{1}{2}u_{rr}\sigma_r^2\left(r_t, t\right) + u_{Vr}\sigma_V\left(V_t, t\right)\sigma_r\left(r_t, t\right)\rho V_t + u_t, \tag{5.15}$$

where ρ is correlation coefficient between ω^* and $\widetilde{\omega}$,

$$\sigma_{x,V}(t) = u_V \sigma_V(V_t, t) V_t \quad and \tag{5.16}$$

$$\sigma_{x,r}(t) = u_r \sigma_r(r_t, t). \tag{5.17}$$

By assumption (5.3), we also have that

$$dV_t + k(V_t, r_t, t) V_t dt = V_t [r_t dt + \sigma_V(V_t, t) d\omega_t^*]. \tag{5.18}$$

We then take $\phi^0 = -1$, because we wish to replicate just one default claim, and substitute (5.2), (5.14) and (5.18) into (5.12) to obtain

$$-u_x(t) dt - \sigma_{x,v}(t) d\omega_t^* - \sigma_{x,r}(t) d\widetilde{\omega}_t - c(V_t, r_t, t) dt + \phi_t^1 V_t [r_t dt + \sigma_V(V_t, t) d\omega_t^*]$$

$$+\phi_t^2 B(t,T) [r_t dt + \sigma_B(r_t, t, T) d\widetilde{\omega}_t] - r_t \left[-X^d(t,T) + \phi_t^1 V_t + \phi_t^2 B(t,T)\right] dt = 0, \tag{5.19}$$

because we replicate the default claim in (5.9), in (5.10) dU_t should be zero without any uncertainty, which means the martingale components and the dt term should be zero. The martingale components, $d\omega_t^*$ and $d\widetilde{\omega}_t$, vanish, so

$$-\sigma_{x,v}(t) + \phi_t^1 V \sigma_V(V_t, t) = 0 \tag{5.20}$$

$$-\sigma_{x,r}(t) + \phi_t^2 B(t,T) \sigma_B(r_t, t, T) = 0, \tag{5.21}$$

and

$$\phi_t^1 V \sigma_V(V_t, t) = \sigma_{x,v}(t) = u_v \sigma_v(V_t, t) V_t \tag{5.22}$$

$$\phi_t^2 B(t,T) \sigma_B(r_t, t, T) = \sigma_{x,r}(t) = u_r \sigma_r(r_t, t). \tag{5.23}$$

The dt term should also be zero, so

$$[u_x(t) + c(V_t, r_t, t) - r_t u(V_t, r_t, t)] dt = 0 \tag{5.24}$$

Substituting (5.15) into (5.24), we have the fundamental PDE,

$$u_t + (r_t - k(V_t, r_t, t)) V_t u_v + \mu_r(r_t, t) u_r + \frac{1}{2}\sigma_v^2(V_t, t) V_t^2 u_{vv}$$

$$+\frac{1}{2}\sigma_r^2(r_t, t) u_{rr} + \sigma_v(V_t, t) \sigma_r(r_t, t) \rho V_t u_{vr} + c(V_t, r_t, t) - r_t u = 0. \tag{5.25}$$

5.2 Pricing defaultable bond with stochastic interest rate models

After deriving the fundamental PDE, we review the previous study of defaultable bonds with a stochastic interest rate.

5.2.1 The model of Kim et al.(1993)

Kim et al. [12] price defaultable bonds with the assumption

$$dr_t = (a - br_t) dt + \sigma_r \sqrt{r_t} d\tilde{w}_t \quad (the\ CIR\ model\ in\ Sec.\ (4.2)). \tag{5.26}$$

$$dV_t = V_t [(r - k) dt + \sigma_V dw_t^*] \tag{5.27}$$

They also assume that the bond's indenture provisions prohibit the company from selling the firm's assets to pay dividends. If we let c be the coupon and k be the payout ratio, then $V_t k$ should always greater than c, because $V_t k$ is the net cash outflow of optimal decisions from the firm. In other words, $\bar{v} = \frac{c}{k}$ is the breakeven point. If $V_t = \bar{v}$ or $V_t k = c$, the dividends match the coupon payment due exactly. If $V_t < \bar{v}$, the dividends are insufficient to cover the coupon payment due. By (5.25), Kim et. al. have the PDE

$$u_t + (r_t - k) V u_V + (a - br) u_r + \frac{1}{2}\sigma_V^2 V^2 u_{VV} \tag{5.28}$$

$$+\frac{1}{2}\sigma_r^2 r u_{rr} + \sigma_V \sigma_r \sqrt{r} V \rho u_{Vr} + c - ru = 0, \tag{5.29}$$

with boundary conditions,

$$u(\bar{v}, r, t) = min(\bar{v}, \delta(T - t) B(t, T, r)), \tag{5.30}$$

where δ represents time varying recovery rate.

$$\lim_{V \to \infty} u(V, r, t) = B(t, T, r), \tag{5.31}$$

and terminal condition,

$$u(V, r, t) = min(V, L). \tag{5.32}$$

Kim et al. only discuss the case when $\bar{v}$ is constant. They solve the PDE with an alternating direction implicit method. In their conclusion, the analysis indicates that their model implies that the credit spreads are close to zero for bonds of short maturities.

5.2.2 The model of Longstaff and Schwartz (1995)

Like Kim et al. [12], Longstaff and Schwartz [14] also price defaultable bonds, but they have different assumptions:

$$dr_t = (\alpha - \beta r_t) dt + \eta d\tilde{\omega}_t \quad (the\ Vasicek\ model\ in\ Sec.\ (4.1)) \tag{5.33}$$

$$dV_t = V_t [r_t dt + \sigma_V d\omega_t^*]. \tag{5.34}$$

They assume that the default occurs when the firm's value is lower than a constant threshold $\bar{v}$. Thus, by (5.25) we have

$$u_t + (r_t - k) V u_V + (\alpha - \beta r) u_r + \frac{1}{2}\sigma_V^2 V^2 u_{VV}$$

$$+\frac{1}{2}\eta^2 u_{rr} + \sigma_V \eta \rho V u_{Vr} + c - ru = 0 \tag{5.35}$$

with boundary conditions

$$u(\bar{v}, r, t) = (1 - w) B(t, T, r) L \tag{5.36}$$

where $(1 - w)$ is the recovery rate and

$$\lim_{V \to \infty} u(V, r, t) = LB(t, T, r), \tag{5.37}$$

Longstaff and Schwartz [14] do not solve the PDE (5.35). They focus on the probabilistic representation of the price of the defaultable bond. If the value of a risky discount bond is $u(X, r, T)$, then

$$u(X, r, T) = B(r, T)[1 - wQ(X, r, T)], \tag{5.38}$$

where $X = \frac{V}{K}$ and K is the default threshold, $B(r, T)$ is the price of default-free zero-coupon bond, and Q=Pr$\{\tau < T | \mathcal{F}_t\}$=Pr{The first passage time of $\ln X$ to zero is less than T}. By Vasicek's model,

$$B(r, T) = \exp(A(T) - D(T) r) \tag{5.39}$$

where

$$A(T) = \left(\frac{\eta^2}{2\beta^2} - \frac{\alpha}{\beta}\right) T + \left(\frac{\eta^2}{\beta^3} - \frac{\alpha}{\beta^2}\right) (\exp(-\beta T) - 1) \tag{5.40}$$

$$- \left(\frac{\eta^2}{4\beta^3}\right) (\exp(-2\beta T) - 1) \tag{5.41}$$

$$D(T) = \frac{1 - \exp(-\beta T)}{\beta} \tag{5.42}$$

Also, by the PDE (5.35), (5.38) and differentiation, $Q(X, r, T)$ is the solution to

$$\frac{\sigma_V^2}{2} X^2 Q_{XX} + \rho\sigma_V \eta X Q_{Xr} + \frac{\eta^2}{2} Q_{rr}$$
$$+ (r - \rho\sigma_V \eta D(T)) X Q_X + \left(\alpha - \beta r - \eta^2 D(T)\right) Q_r - Q_T = 0. \tag{5.43}$$

Using the result in Friedman [9], we know

$$d \ln X = r - \frac{\sigma^2}{2} - \rho\sigma\eta D(T - t) \, dt + \sigma d\omega_1 \tag{5.44}$$

$$dr = \left(\alpha - \beta r - \eta^2 D(T - t)\right) dt + \eta d\omega_2 \tag{5.45}$$

Integrating the dynamics for r from zero to τ, gives

$$r_\tau = r \exp(-\beta\tau) + \left(\frac{\alpha}{\beta} - \frac{\eta^2}{\beta^2}\right) (1 - \exp(\beta\tau))$$

$$+ \frac{\eta^2}{2\beta^2} \exp(-\beta T) (\exp(\beta\tau) - \exp(-\beta\tau)) + \eta \exp(-\beta\tau) \int_0^\tau \exp(\beta s) \, d\omega_2 \tag{5.46}$$

Integrating for $\ln(X)$, substituting for the value of r from (5.46), and by applying Fubini's Theorem, we have

$$\ln X_T = \ln X + M(T, T) + \frac{\eta}{\beta} \int_0^\tau 1 - \exp(-\beta(T - t)) \, d\omega_2 + \sigma \int_0^\tau d\omega_1, \tag{5.47}$$

where

$$M(t, T) = \left(\frac{\alpha - \rho\sigma\eta}{\beta} - \frac{\eta^2}{\beta} - \frac{\sigma^2}{2}\right) t + \left(\frac{\rho\sigma\eta}{\beta^2} - \frac{\eta^2}{2\beta^3}\right) \exp(-\beta T) (\exp(\beta t) - 1)$$
$$+ \left(\frac{r}{\beta} - \frac{\alpha}{\beta^2} + \frac{\eta^2}{\beta^3}\right) (1 - \exp(-\beta t)) - \frac{\eta^2}{2\beta^3} \exp(-\beta T) (1 - \exp(\beta t)) \tag{5.48}$$

and

$$S(t) = \left(\frac{\rho\sigma\eta}{\beta} + \frac{\eta^2}{\beta^2} + \sigma^2\right) t - \left(\frac{\rho\sigma\eta}{\beta^2} - \frac{2\eta^2}{\beta^3}\right) (1 - \exp(-\beta t)) + \frac{\eta^2}{2\beta^3} (1 - \exp(-2\beta t)) \tag{5.49}$$

By (5.47), $\ln X_T$ is normally distributed with mean $\ln X + M(T, T)$, the variance $S(T)$ is similar, so

$$\ln X_T \mid \ln X_t = 0 \sim N(M(T, T) - M(t, T), S(T) - S(t)). \tag{5.50}$$

Let $q(0, \tau \mid \ln X, 0)$ be the first passage density of $\ln X$ at time zero. By Buonocore et al. [7],

$$N\left(\frac{-\ln X - M(t, T)}{\sqrt{S(t)}}\right) = \int_0^t q(0, \tau \mid \ln X, 0) \, N\left(\frac{M(\tau, T) - M(t, T)}{\sqrt{S(t) - S(\tau)}}\right) d\tau. \tag{5.51}$$

Longstaff and Schwartz [14] assume that $q(0,\tau \mid \ln X, 0)$ is a constant on each time interval $[(i-1)T/n, iT/n]$, and so they rewrite (5.51) by

$$N(a_i) = \sum_{j=1}^{i} q_i N(b_{ij}) \tag{5.52}$$

where

$$q_i = q\left(0, \frac{iT}{n} \mid \ln X, 0\right)\frac{T}{n}, \tag{5.53}$$

$$a_i = \frac{-\ln X - M\left(\frac{iT}{n}, T\right)}{\sqrt{S\left(\frac{iT}{n}\right)}}, \tag{5.54}$$

$$b_{ij} = \frac{M\left(\frac{jT}{n}, T\right) - M\left(\frac{iT}{n}, T\right)}{\sqrt{S\left(\frac{iT}{n} - \frac{jT}{n}\right)}}. \tag{5.55}$$

and

$$Q(X, r, T, n) = \sum_{i=1}^{n} q_i \tag{5.56}$$

where

$$q_1 = N(a_1) \tag{5.57}$$

$$q_i = N(a_i) - \sum_{j=1}^{i-1} q_j N(b_{ij}), \tag{5.58}$$

and

$$\lim_{n\to\infty} Q(X, r, T, n) = Q(X, r, T). \tag{5.59}$$

Longstaff and Schwartz's model [14] has following good features:

1. The correlation of a firm's assets with changes in the level of the interest rate can have significant effects on the price of defaultable bonds.
2. The term structure of credit spreads can fit a variety of data.
3. The model implies that credit spreads are negatively related to the level of the interest rate. In other words, the credit spread is larger when the interest rate is at a lower level.

5.2.3 The model of Briys and de Varenne (1997)

Briys and de Varenne [6] assume that

$$dr_t = a(t)(b(t) - r_t)dt + \sigma(t)d\tilde{\omega}_t \quad (the\ Hull - White\ model) \tag{5.60}$$

$$dV_t = V_t\left[r_t dt + \sigma_V\left(\rho d\tilde{\omega}_t + \sqrt{1-\rho^2}d\omega_t^*\right)\right] \tag{5.61}$$

with the default threshold

$$v_t = L\ when\ t = T, \tag{5.62}$$

$$v_t = KB(t,T)\ when\ t < T, \tag{5.63}$$

and boundary conditions

$$u(V,T) = \beta_1 L\ when\ t = T\ and\ K < V < L, \tag{5.64}$$

$$u\left(v_t,t\right)=\beta_2 v_t\, when\, t<T\, and\, V=v_t. \tag{5.65}$$

The parameters satisfy $0<K<L$, $0<\beta_2<1$ and $0<\beta_2<1$, where K is a quantity given in the safety covenant. If K is large, the bond buyer is highly protected. The bond is easy to default, but the bond buyer can receive almost as much as L when the default occurs. The β_1 and β_2 are the recovery rate for time $t=T$ and $t<T$ respectively. $B\left(t,T\right)$ is a default-free bond price.

Briys and de Varenne's model [6] is Black and Cox's model [2] with a stochastic interest rate, and they allow the value of the firm's assets and interest rate to be correlated. Here, the default threshold is not a constant anymore; it is a function of t instead. For that reason, the location of the left boundary is also a function of t, which means this model has a moving boundary.

For the interest rate Briys and de Varenne use the generalized Vasicek model, not the CIR model [8], because under the generalized Vasicek model the volatility of the default-free bond price is just a deterministic function. Thus, there is an analytic solution for the forward price $F_D\left(t,T\right)=\frac{D(t,T)}{B(t,T)}$, where $D\left(t,T\right)$ is a defaultable bond price.

$$F_D\left(t,T\right)=L-D_1\left(t,T\right)+D_2\left(t,T\right)-\left(1-\beta_2\right)\left[F_tN\left(d_4\right)+KN\left(d_3\right)\right] \tag{5.66}$$

$$-\left(1-\beta_1\right)F_t\left[N\left(d_2\right)-N\left(d_4\right)\right]-\left(1-\beta_1\right)K\left[N\left(d_5\right)-N\left(d_3\right)\right] \tag{5.67}$$

where

$$F_t=\frac{V_t}{B(t,T)} \tag{5.68}$$

$$D_1\left(t,T\right)=LN\left(d_1\right)-F_tN\left(d_2\right) \tag{5.69}$$

$$D_2\left(t,T\right)=KN\left(d_5\right)-\frac{F_tL}{K}N\left(d_6\right) \tag{5.70}$$

$$d_1=\frac{ln\left(\frac{L}{F_t}\right)+\frac{1}{2}\sigma^2\left(t,T\right)}{\sigma\left(t,T\right)}=d_2+\sigma\left(t,T\right) \tag{5.71}$$

$$d_3=\frac{ln\left(\frac{K}{F_t}\right)+\frac{1}{2}\sigma^2\left(t,T\right)}{\sigma\left(t,T\right)}=d_4+\sigma\left(t,T\right) \tag{5.72}$$

$$\sigma^2\left(t,T\right)=\int_t^T\left[\rho\sigma_V-b\left(u,T\right)\right]^2+\left(1-\rho^2\right)\sigma_V^2du. \tag{5.73}$$

This model can fit quite diverse data in the term structure of the credit spread. (Recall that the credit spread is the difference between the default-free interest rate and the yield of the defaultable bond.) However there is a drawback. Under Briys and Varenne's assumption, the coupon payment c to the holder of the bonds could be greater than the firm's value before the default occurs, since in the model, the coupon payment is independent of the stochastic barrier and of the firm's value. If the V_t and the $B\left(t,T\right)$ both drop to the extremely low levels such that $V_t<LB\left(t,T\right)$, V_t may be less than c.

5.2.4 The model of Saa-Requejo et al.(1999)

Saa-Requejo et al.[20] assume that

$$dr_t=\mu_rdt+\sigma_rd\tilde{\omega}_t \tag{5.74}$$

$$dV_t=V_t\left[\left(r_t-k\right)dt+\sigma_Vd\omega_t^*\right] \tag{5.75}$$

$$\omega_t^* = \rho\tilde{\omega}_t + \sqrt{1-\rho^2}\hat{\omega}_t \tag{5.76}$$

$$dv_t = v_t\left[(r_t - \zeta)\,dt + \tilde{\sigma}_v d\tilde{\omega}_t + \hat{\sigma}_v d\hat{\omega}_t\right]. \tag{5.77}$$

We can see that in this model, the value of each liability of the firm is a stochastic process. Saa-Requejo et al. derive a general 3-D PDE for this model. However they can not find the probabilistic representation of the price of the defaultable bond anymore. Saa-Requejo et al. only discuss some special cases that are solvable.

In order to derive the 3-D PDE, one considers a self-financing trading strategy,

$$\phi_t = \left(\phi_t^0, \phi_t^1, \phi_t^2, \phi_t^3\right), \tag{5.78}$$

$$U_t(\phi) = \phi_t^0 u\left(V_t, v_t, B(t,T), t\right) + \phi_t^1 V_t + \phi_t^2 v_t + \phi_t^3 B(t,T) \tag{5.79}$$

$$\begin{aligned} dU_t(\phi) &= \phi_t^0\left[du\left(V_t, v_t, B(t,T), t\right) + c\left(V_t, v_t, B(t,T)\right)dt\right] \\ &+\phi_t^1\left[dV_t + kV_t dt\right] + \phi_t^2\left[dv_t + \zeta v_t dt\right] + \phi_t^3 dB(t,T). \end{aligned} \tag{5.80}$$

The Itô differential $du\left(V_t, v_t, B(t,T), t\right)$ is

$$\begin{aligned} du &= u_t dt + u_V dV_t + u_v dv_t + u_B dB(t,T) + u_{Vv} d < V, v >_t \\ &+u_{VB} d < V, B >_t + u_{Bv} d < B, v >_t + \frac{1}{2}u_{VV} d < V, V >_t \\ &+\frac{1}{2}u_{vv} d < v, v >_t + \frac{1}{2}u_{BB} d < B, B >_t \end{aligned} \tag{5.81}$$

where

$$d < V, V >_t = \sigma_V^2 V_t^2 dt, \tag{5.82}$$

$$d < v, v >_t = \left(\tilde{\sigma}_v^2 + \hat{\sigma}_v^2\right) dt, \tag{5.83}$$

$$d < B, B >_t = b^2(t,T)\, B^2(t,T)\, dt, \tag{5.84}$$

$$d < V, v >_t = \sigma_V\left(\rho\tilde{\sigma_v} + \sqrt{1-\rho^2}\hat{\sigma}_v\right) v_t V_t dt, \tag{5.85}$$

$$d < V, B >_t = \sigma_V \rho V_t b(t,T)\, B(t,T)\, dt, \tag{5.86}$$

$$d < B, v >_t = \tilde{\sigma}_V v_t b(t,T)\, B(t,T)\, dt. \tag{5.87}$$

Because dU_t should be zero without any uncertainty, the martingale compenents vanish, in (5.80), the coefficients of $d\tilde{\omega}_t$, $d\hat{\omega}_t$ and $d\omega_t^*$ should be zeros, and

$$\phi_t^1 = u_V,\ \phi_t^2 = u_v,\ \phi_t^3 = u_B. \tag{5.88}$$

Substituting (5.88) into (5.80),

$$du = \phi_t^1\left[dV_t + kV_t dt\right] + \phi_t^2\left[dv_t + \zeta v_t dt\right] + \phi_t^3 dB(t,T) - cdt \tag{5.89}$$

$$= u_V\left[dV_t + kV_t dt\right] + u_v\left[dv_t + \zeta v_t dt\right] + u_B dB(t,T) - cdt. \tag{5.90}$$

Then the coefficient of dt leads to the PDE,

$$\begin{aligned} &u_t + kVu_V + \zeta v u_v + \sigma_V\left(\rho\tilde{\sigma_v} + \sqrt{1-\rho^2}\hat{\sigma}_v\right) vVu_{Vv} + \sigma_V \rho bVBu_{VB} + \tilde{\sigma}_v vbBu_{Bv} \\ &+\frac{1}{2}\sigma_V^2 V^2 u_{VV} + \frac{1}{2}v^2\left(\tilde{\sigma}_v^2 + \hat{\sigma}_v^2\right)u_{vv} + \frac{1}{2}b^2 B^2 u_{BB} = 0. \end{aligned} \tag{5.91}$$

Saa-Requejo et al.[20] compare solutions of (5.91) with empirical literature, and error of pricing is low under their model. In addition, the results do not suffer from the pricing biases observed by contemporary empirical studies on Longstaff and Schwartz's model [14]. The reason is that the solvency ratio, $X = \frac{V}{v}$, follows

$$dX(t) = \mu_x dt + \sigma_x d\omega_x, \tag{5.92}$$

where the drift term, μ_x, is not a function of interest rate r under Saa-Requejo's model. When $X < 1$ we say that the default event occurs. Thus, under Saa-Requejo's model the probability of default and the interest rate are independent. In the other hand, in Longstaff and Schwartz's model the default threshold, v, is constant, so the drift term of X is a increasing function of r. Thus, under Longstaff and Schwartz's model, when r goes up, the probability of default goes down which does not fit empirical studies.

Chapter 6

Pricing CDS and Future work

In Chapter 5 we showed how to price defaultable bonds. In this chapter we introduce a new method to price the most popular credit derivative in the market, credit default swap (CDS) in the same framework. The only difference new in the framework is that we use the BDT model [3] for the interest rate, which has the advantage mentioned in Sec. 4.4. There is no analytic solution for the PDE with the BDT model however. For that reason, we will have to solve the PDE with numerical methods. We discuss how to derive the new pricing PDE form the fundamental PDE (5.25) and the numerical issues of the new method in following sections.

6.1 Pricing CDS with PDE methods

The CDS is a kind of insurance that protects the buyer of the CDS when a default event occurs. As a traditional insurance, the protection buyer makes regular premium payments quarterly or semiannually. When the default event occurs, the protection seller pays par value, L, of the bond to the buyer, and the buyer physically delivers the bond to the seller. The buyer would then cease paying premiums. Thus, to the protection seller's point of view, the price of a CDS, $u(V_t, r_r, t)$, is the same as a defaultable bond with coupon payment c which is the premium payment, and the defaultable bond has face value that equals zero. When the default event occurs, the protection seller pays par value minus the rest value of the defaultable bond.

For our CDS model we take the default threshold to be the same as in Black and Cox's model [2],

$$v_t = KB(t,T) \; when \, t < T \tag{6.1}$$

$$v_t = L \, when \, t = T. \tag{6.2}$$

The boundary conditions become

$$u(v_t, r_r, t) = v_t - L, \tag{6.3}$$

$$\lim_{V_t \to \infty} u(V_t, r_r, t) = 0. \tag{6.4}$$

The default-free bond price $B(t,T)$ in the boundary condition (6.1) also follows a PDE when r_t is stochastic. It is the price of the default-free zero coupon bond. In order to derive the general PDE, Neftci

[19] assumes that

$$dr_t = \mu\left(r_t, t\right)dt + \sigma\left(r_t, t\right)d\omega_t. \tag{6.5}$$

By Itô's lemma,

$$dB = \left(\frac{\partial B}{\partial t} + \frac{\partial B}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2\right)dt + \frac{\partial B}{\partial r}\sigma d\omega_t. \tag{6.6}$$

We let $\mu_B\left(t, T\right) = \frac{\partial B}{\partial t} + \frac{\partial B}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2$ and $\nu_B\left(t, T\right) = \frac{\partial B}{\partial r}\sigma$. We can only use a zero coupon bond with a different maturity to create a hedge portfolio, so

$$\pi_t = -B\left(t, T\right) + \triangle B\left(t, T^*\right)\ \ T^* > T \tag{6.7}$$

and

$$d\pi_t = -dB\left(t, T\right) + \triangle dB\left(t, T^*\right) \tag{6.8}$$

$$= \left(\triangle\mu_B\left(t, T^*\right) - \mu_B\left(t, T\right)\right)dt + \left(-\frac{\partial B\left(t, T\right)}{\partial r}\sigma + \triangle\frac{\partial B\left(t, T^*\right)}{\partial r}\sigma\right)d\omega_t. \tag{6.9}$$

Because π_t is risk-free, the coefficient of $d\omega_t$ should be zero,

$$-\frac{\partial B\left(t, T\right)}{\partial r}\sigma + \triangle\frac{\partial B\left(t, T^*\right)}{\partial r}\sigma = 0. \tag{6.10}$$

Solving for $\triangle$, we have

$$\triangle = \frac{\frac{\partial B(t,T)}{\partial r}\sigma}{\frac{\partial B(t,T^*)}{\partial r}\sigma} = \frac{\nu_B\left(t, T\right)}{\nu_B\left(t, T^*\right)}. \tag{6.11}$$

Substituting (6.11) and $d\pi_t = r_t\pi_t dt$ into (6.9),

$$\left(\frac{\nu_B\left(t, T\right)}{\nu_B\left(t, T^*\right)}\mu_B\left(t, T^*\right) - \mu_B\left(t, T\right)\right)dt = r_t\pi_t dt \tag{6.12}$$

$$= r_t\left(-B\left(t, T\right) + \frac{\nu_B\left(t, T\right)}{\nu_B\left(t, T^*\right)}B\left(t, T^*\right)\right)dt. \tag{6.13}$$

Thus,

$$\frac{\nu_B\left(t, T\right)}{\nu_B\left(t, T^*\right)}\mu_B\left(t, T^*\right) - \mu_B\left(t, T\right) = r_t\left(-B\left(t, T\right) + \frac{\nu_B\left(t, T\right)}{\nu_B\left(t, T^*\right)}B\left(t, T^*\right)\right). \tag{6.14}$$

We can rewrite (6.14) with notation λ_t,

$$\frac{\mu_B\left(t, T\right) - r_t B\left(t, T\right)}{\nu_B\left(t, T\right)} = \frac{\mu_B\left(t, T^*\right) - r_t B\left(t, T^*\right)}{\nu_B\left(t, T^*\right)} \equiv \lambda_t. \tag{6.15}$$

Eq. (6.15) means that market price of risk does not depend on maturity date.

By solving for $\mu_B\left(t, T\right)$ from (6.15),

$$\mu_B\left(t, T\right) = r_t B\left(t, T\right) + \lambda_t\nu_B\left(t, T\right), \tag{6.16}$$

and by definition of $\mu_B\left(t, T\right)$,

$$\frac{\partial B}{\partial t} + \frac{\partial B}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2 = r_t B\left(t, T\right) + \lambda_t\nu_B\left(t, T\right) = r_t B + \lambda_t\frac{\partial B}{\partial r}\sigma. \tag{6.17}$$

For the BDT model [3], we just substitute $\mu = r_t\left(a\left(t\right)\left(b\left(t\right) - ln\left(r_t\right)\right) + \frac{1}{2}\eta_t^2\right)$ and $\sigma = \eta_t r_t$ into (6.17). Thus, our PDE for $B\left(t, T\right)$ under BDT model is

$$\frac{\partial B}{\partial t} + \frac{\partial B}{\partial r}\left[r\left(a\left(t\right)\left(b\left(t\right) - ln\left(r\right)\right) + \frac{1}{2}\eta_t^2\right) - \lambda_t \eta_t r\right] + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\left(\eta_t r\right)^2 = rB. \tag{6.18}$$

In order to have the CDS pricing PDE, we substitute the coefficients of the BDT model [3] in (4.6) into the fundamental PDE (5.25), we have the new CDS pricing PDE,

$$u_t + \left(r_t - k\right)Vu_v + r_t\left(a\left(t\right)\left(b\left(t\right) - lnr_t\right) + \frac{1}{2}\eta_t^2\right)u_r + \frac{1}{2}\sigma_v^2 V^2 u_{vv}$$

$$+\frac{1}{2}r_t^2\eta_t^2 u_{rr} + \sigma_v \eta_t r_t \rho V u_{vr} + c - r_t u = 0. \tag{6.19}$$

The terminal conditions are

$$u\left(V_t, r_r, t\right) = 0 \quad when \quad V_t > L, \tag{6.20}$$

and

$$u\left(V_t, r_r, t\right) = \beta V_t - L \quad when \quad K < V_t < L, \tag{6.21}$$

where β is the recovery rate.

The boundary conditions are

$$u\left(v_t, r_r, t\right) = v_t - L, \tag{6.22}$$

$$\lim_{V_t \to \infty} u\left(V_t, r_r, t\right) = 0, \tag{6.23}$$

where

$$v_t = KB\left(t, T\right)\ when\, t < T \tag{6.24}$$

$$v_t = L\, when\, t = T, \tag{6.25}$$

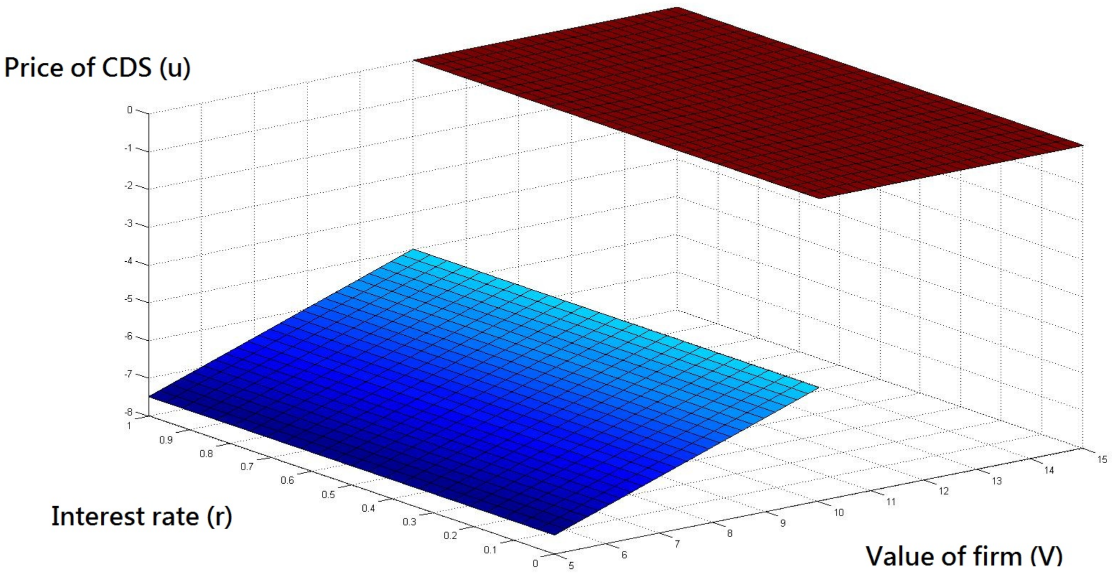
and $B\left(t, T\right)$ follows 6.18.

6.2 Numerical issues and future work

After involving the stochastic interest rate, we can not only price a defaultable bond but also a CDS. In Sec. 6.1, we derived the PDE for pricing a CDS. It is difficult to find the analytic solution for the PDE, so we have to solve it numerically. By the discussion in Sec. 6.1, we can see that there are some numerical issues which we will face when solving it.

First, the payoff of CDS at the maturity date may not be continuous. Recalling the definition of CDS, the protection buyer receives nothing if no default event occurs; the protection buyer receives face value of the bond and physically delivers the bond to the protection seller if a default event occurs. From the protection seller's point of view, if $V > L$ at maturity date, the payoff is L; if $K < V < L$ at maturity date, the payoff

Figure 6.1: The initial condition of the CDS pricing PDE with $L = 10$, $K = 5$ and $\beta = 0.5$.



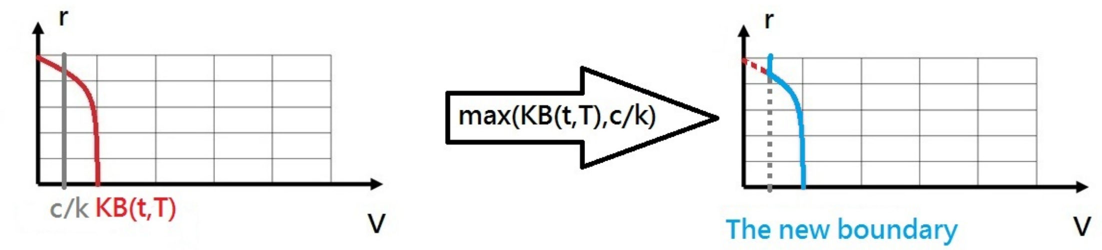
is $\beta V_t - L$. For that reason, we can see that the payoff is not continuous at $V = L$ at the maturity date when β is less than 1 in Fig. 6.1.

Another issue is how to choose the maximum value of r_t, r_{max}. In the models of the interest rate, r_t could be large with small probability. In order to solve the problem, we need to choose the maximum value of r_t. The maximum value of r_t maybe influence the price of bond when we approximate the solution. If the advection term is negative, the solution moves to left (the direction of r decreasing). In other words, the large maximum value of r_t is, the faster the speed of advection; the large maximum value of r_t is, the more mesh we need, because we only care about the price when r_t is small. It is a trade-off.

Furthermore, the right boundary condition in (6.18) is another issue. We can easily set $B(r_{max}, t) = 0$ and set r_{max} to be extremely high, but there are several disadvantages we discuss in last paragraph. Another choice is $B(r_{max}, t) = e^{-r_{max}(T-t)}$, but if r_{max} is not large enough, it still can influence the price $B(r, t)$.

After we can handle the issues above, we can consider the more complicated boundary condition. Recall that the default threshold $v = KB(t, T)$ has a main issue that it allows the value of the firm lower than dividend of the bond which the firm need to pay at the end of every period. So if we want avoid this situation, we can use the threshold which Kim et al. use in Sec. 5.2.1. We can set the default threshold as $v = max(KB(t, T), \frac{c}{k})$ which is sketched in Fig. 6.2. With the new threshold, we cannot use the changing variable approach which we used in Chapter 3, so it is a new challenge. We need to figure out a new approach to solve it.

Figure 6.2: The new boundary for future work.



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