Pricing A Mortgage-backed Security Using First Hitting Time

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Motivation

- At each point in time during the amortization period, the borrower has the right to choose among four payment actions: pay the scheduled payment, a complete prepayment of the loan, default, or curtailment.
- The borrowers prepay if the interest rate is low and some borrowers sell their house when the housing price is high.
- The borrowers will default if the housing price is lower than the unpaid balance

- All of above involve a first hitting time problem
- we will use first hitting time to price a mortgage-backed security.

Literature Review

- Lo, Chung and Hui (2007) propose a simple and easy-to-use method for computing an accurate estimate of the double barrier hitting time distribution of a mean-reverting lognormal process, and discuss its application to pricing exotic options whose payoffs are contingent upon
- Lo and Hui (2006) derive the closed-form formula for the first passage time density of a time-dependent Ornstein-Uhlenbeck process.
- Barthélémy and Prigent(2009) derived the optimal time to sell the real estate portfolio to maximize the net present value of free cash flow of housing.

Model Setup

- Stochastic process for housing and Interest rate

- $$dH_t = \mu_h H_t dt + \sigma_h H_t dW_t \quad (1)$$

  - μ_r is the expected growth rate of the house price,
  - σ_h is the volatility of house price growth rate, and W_t is a standard Brownian motion process.

$$dr_t = \theta(\mu_r - r_t)dt + \sigma_r dW_{1t}, \qquad (2)$$

μ_r : the mean reversion level, which is the weighted average level of the interest rate process

θ : the mean reversion speed

σ_r : the volatility of the interest rate

W_{1t} : the standard Brownian motion process which is independent with W_t

First Hitting Time

- Denoted τ_r is the first hitting time to hit the lower barrier (r_l) of interest rate. ie

$$\tau_r = \inf\{t; r(t) = r_l, r_l < r(s) \ \ for\ all\ s \in [0,t)\}$$

- Since the interest rate follows OU process, the distribution function of time hitting is r_0

- $$P_{fp}(r_0,t) = \Phi\left(\frac{2\beta\eta(t)+r_0}{\sqrt{2\eta(t)}}\right) + \Phi(-\frac{2\beta\eta(t)-r_0}{\sqrt{2\eta(t)}})e^{-2\beta r_0} \quad (5)$$

where $\alpha(t) = -\int_0^t u(t')dt' = -\int_0^t -\theta dt' = \theta t$

$$\gamma(t) = -\int_0^t v(t')e^{\alpha(t')}dt' = -\int_0^t \theta\mu_r e^{\theta t'}dt' = \mu_r\left(1 - e^{\theta t}\right)$$

$$\eta(t) = \int_0^t \frac{1}{2}\sigma_r{}^2 e^{2\alpha(t')}dt' = \int_0^t \frac{1}{2}\sigma_r{}^2 e^{2\theta t'}dt' = \frac{\sigma_r{}^2}{4\theta}(e^{2\theta t} - 1)$$

$$\beta = -\frac{\int_0^\tau \gamma(t)\eta(t)e^{-2\alpha(t)}dt}{2\int_0^\tau \eta^2(t)e^{-2\alpha(t)}dt}$$

- Denote τ_L and τ_U as the first hitting time to the lower barrier (without hitting the upper barrier earlier) and the first hitting time to the upper barrier (without hitting the lower barrier earlier), respectively, i.e.,

$$\tau_l = \inf\left\{t; H\left(t\right) = L, L < H\left(s\right) < U \ \ for\, all\, s \le t\right\}$$

$$\tau_u = \inf\left\{t; H\left(t\right) = U, L < H\left(s\right) < U \ \ for\, all\, s \le t\right\}$$

- The cumulative density function of τ_L and τ_H are

$$G_L(t;S,L,U)=\left[\frac{L}{S}\right]^{\left(\mu_h\sigma_h^{-2}-1/2\right)} *$$

$$\left\{\sum_{n=0}^{\infty}\left[e^{-a_n(\mu_h\sigma_h^{-1}-\frac{\sigma_h}{2})}\Phi\left(\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t-\sigma_h a_n}{\sigma_h\sqrt{t}}\right)+e^{a_n(\mu_h\sigma_h^{-1}-\frac{\sigma_h}{2})}\Phi\left(-\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t+\sigma_h a_n}{\sigma_h\sqrt{t}}\right)\right]-\sum_{n=-\infty}^{-1}\left[e^{a_n(\mu_h\sigma_h^{-1}-\frac{\sigma_h}{2})}\Phi\left(\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t+\sigma_h a_n}{\sigma_h\sqrt{t}}\right)+e^{-a_n(\mu_h\sigma_h^{-1}-\frac{\sigma_h}{2})}\Phi\left(-\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t-\sigma_h a_n}{\sigma_h\sqrt{t}}\right)\right]\right\}$$

$$G_U(t;S,L,U)=\left[\frac{U}{S}\right]^{\left(\mu_h\sigma_h^{-2}-1/2\right)} *$$

$$\left\{\sum_{n=0}^{\infty}\left[e^{-b_n(\mu_h\sigma_h^{-1}-\frac{\sigma_h}{2})}\Phi\left(\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t-\sigma_h b_n}{\sigma_h\sqrt{t}}\right)+e^{b_n(\mu_h\sigma_h^{-1}-\frac{\sigma_h}{2})}\Phi\left(-\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t+\sigma_h b_n}{\sigma_h\sqrt{t}}\right)\right]-\sum_{n=-\infty}^{-1}\left[e^{b_n(\mu_h\sigma_h^{-1}-\frac{\sigma_h}{2})}\Phi\left(\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t+\sigma_h b_n}{\sigma_h\sqrt{t}}\right)+e^{-b_n(\mu_h\sigma_h^{-1}-\frac{\sigma_h}{2})}\Phi\left(-\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t-\sigma_h b_n}{\sigma_h\sqrt{t}}\right)\right]\right\}$$

- where

$$a_n = \frac{1}{\sigma_h} ln \frac{U^{2n} S}{L^{2n+1}};$$

$$b_n = \frac{1}{\sigma_h} ln \frac{U^{2n+1}}{L^{2n} S}$$

$\Phi(.)$: the cumulative density function of standard normal dustribution

S: Initial housing price

Cash flow at time t

As H_t hits U at t before L; real estate investors will sell the house and prepay, the cash flow at time t is

$$CF_{1t} = PMT + UPB_t$$

where PMT is the monthly payment and UPB is the unpaid balance

As r_t hits r_l, borrowers are willing to take on a new loan and repay old loan

$$CF_{2t} = PMT + UPB_t$$

As H_t hits $L = UPB_t$, borrowers will default

$$CF_{3t}^{a} = H_t$$

As H_t and r_t don't hit any barrier

$$CF_{4T} = PMT$$

Adjustment the c.d.f. of hitting time

- Since the lower bound of housing price $L = UPB_t$ is a function of t , which is not a constant, therefore we need to made following adjust for c.d.f. of hitting time

i. $G_U(t_1; S, L_1, U)$ is logical and let $G_U^{adj}(t_1) = G_U(t_1; S, L_1, U)$.

ii. The probability of $G_U(t_2; S, L_2, U)$ is overvalued because of including $L_2 \leq H_1 \leq L_1$ at time t_1. Then borrows should default at time t_1. It is impossible to touch U at time t_2. So, the adjusted G_U is:

$$G_U^{adj}(t_2) = G_U(t_2; S, L_2, U) - [G_U(t_1; S, L_2, U) - G_U(t_1; S, L_1, U)]$$

iii. Similarly, the probability of $G_U(t_3; S, L_3, U)$ needs to include $L_2 \leq H_1 \leq L_1$ at time t_1 and $L_3 \leq H_2 \leq L_2$ at time t_2.

$$G_U^{adj}(t_3) = G_U(t_3; S, L_3, U) - [G_U(t_1; S, L_2, U) - G_U(t_1; S, L_1, U)] - [G_U(t_2; S, L_3, U) - G_U(t_2; S, L_2, U)]$$

Therefore

$$G_U^{adj}(t_n) = G_U(t_n; S, L_n, U) - \sum_{i=1}^{n-1} [G_U(t_i; S, L_{i+1}, U) - G_U(t_i; S, L_i, U)]$$

$$n = 2,3, \dots, T$$

Similarly,

$$G_L^{adj}(t_1) = G_L(t_1; S, L_1, U)$$

$$G_L^{adj}(t_n) = G_L(t_n; S, L_n, U) + \sum_{i=1}^{n-1} [G_U(t_i; S, L_i, U) - G_U(t_i; S, L_{i+1}, U)]$$

$$n = 2,3, \dots, T$$

Pricing a mortgage-backed security

The cumulative probability of hitting a barrier at time t

	H_t hit $U(\tau_u < t)$	No hitting	H_t hit $L(\tau_l < t)$	
No hitting $(\tau_r > t)$	ρ proportion of borrowers making prepayment	Continue holding until next period	default	1-$P_{fp}(r_l, t)$
r_t hit $r_l(\tau_r < t)$	prepayment	prepayment	default	$P_{fp}(r_l, t)$
	$G_U^{adj}(t)$	$1 - G_U^{adj}(t) - G_L^{adj}(t)$	$G_L^{adj}(t)$	1

For example it has probability $G_u^{adj}(t)$ *(1-$P_{fp}(r_l, t)$) $(\tau_r > t)$ and $(\tau_u < t)$, In this situation, ρ proportion of borrowers making prepayment

Define

$$\Delta G_U(t; S, L, U) = G_U^{adj}(t) - G_U^{adj}\left(t - \frac{1}{12}\right), \quad (10)$$

$$\Delta G_L(t; S, L, U) = G_L^{adj}(t) - G_L^{adj}\left(t - \frac{1}{12}\right), \quad (11)$$

$$\Delta P_{fp}(r_0, t) = P_{fp}(r_0, t) - P_{fp}\left(r_0, t - \frac{1}{12}\right), \quad (12)$$

$$P_1(t) \equiv P_{\rho\ \mathrm{prepayment}}(t) = \rho \times \Delta G_U(t; S, L, U) \times [1 - P_{fp}(r_0, t)], \qquad (13)$$

$$P_2(t) \equiv P_{prepayment}(t) = \left[\Delta G_U(t; S, L, U) + 1 - G_U^{adj}(t) - G_L^{adj}(t)\right] \times \Delta P_{fp}(r_0, t)$$

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$$P_3(t) \equiv P_{default}(t) = \Delta G_L(t; S, L, U) \times [1\text{-}P_{fp}(r_0, t) + \Delta P_{fp}(r_0, t)], \qquad (15)$$

$$P_4(T) \equiv P_{holding\ to\ maturity}(T) = 1 - \sum_{t=1/12}^{T-1/12} [P_1(t) + P_2(t) + P_3(t)]. \qquad (16)$$

NPV of each state

$$: PV_{1t} = \text{PMT}*(e^{-\int_0^{\frac{1}{12}} r_s ds} + e^{-\int_0^{\frac{2}{12}} r_s ds} + \ldots\ldots + e^{-\int_0^{t} r_s ds}) + UPB_t$$

$$= \text{PMT}*(e^{-\int_0^{\frac{1}{12}} r_s ds} + e^{-\int_0^{\frac{2}{12}} r_s ds} + \ldots\ldots + e^{-\int_0^{t-\frac{1}{12}} r_s ds}) + H_t * e^{-\int}$$

$$= \text{PMT}*(e^{-\int_0^{\frac{1}{12}} r_s ds} + e^{-\int_0^{\frac{2}{12}} r_s ds} + \ldots\ldots + e^{-\int_0^{t-\frac{1}{12}} r_s ds}) + UPB_t * e$$

Since r_t is stochastic, we need to take expectation over r

$$V_{1t} = E[PV_{1t}] = E[PMT \times (e^{-\int_0^{\frac{1}{12}} r_s ds} + e^{-\int_0^{\frac{2}{12}} r_s ds} + \cdots \ldots + e^{-\int_0^t r_s ds}) + UPB_t \times$$

$$e^{-\int_0^t r_s ds}] = PMT \times \sum_{i=1/12}^{t} E\left(e^{-\int_0^i r_s ds}\right) + UPB_t \times E\left(e^{-\int_0^t r_s ds}\right)$$

$$V_{2t}^a = \mathrm{E}[PV_{2t}^a] = E\left[PMT \times \left(e^{-\int_0^{\frac{1}{12}} r_s ds} + e^{-\int_0^{\frac{2}{12}} r_s ds} + \cdots \ldots + e^{-\int_0^{t-\frac{1}{12}} r_s ds}\right) + H_t \times\right.$$

$$\left. e^{-\int_0^t r_s ds}\right] = PMT \times \sum_{i=\frac{1}{12}}^{t-\frac{1}{12}} E\left(e^{-\int_0^i r_s ds}\right) + H_0$$

$$V_{2t}^b = \mathrm{E}[PV_{2t}^b] = E\left[PMT \times \left(e^{-\int_0^{\frac{1}{12}} r_s ds} + e^{-\int_0^{\frac{2}{12}} r_s ds} + \cdots \ldots + e^{-\int_0^{t-\frac{1}{12}} r_s ds}\right) + UPB_t\right.$$

$$\left. \times\ e^{-\int_0^t r_s ds}\right] = PMT \times \sum_{i=\frac{1}{12}}^{t-\frac{1}{12}} E\left(e^{-\int_0^i r_s ds}\right) + UPB_t \times E\left(e^{-\int_0^t r_s ds}\right)$$

$$V_{3T} = E[PV_{3T}] = E\left[PMT * \left(e^{-\int_0^{\frac{1}{12}} r_s ds} + e^{-\int_0^{\frac{2}{12}} r_s ds} + \cdots \ldots + e^{-\int_0^T r_s ds}\right)\right] = PMT \times$$

$$\sum_{i=\frac{1}{12}}^{T} E\left(e^{-\int_0^i r_s ds}\right)$$

where

$$E\left(e^{-\int_0^t r_s ds}\right) = exp\,(-\mu_r t - (r_0 - \mu_r)\frac{1-e^{-\theta t}}{\theta} + \frac{\sigma_r{}^2}{2\theta^2}\left(t - \frac{1-e^{-\theta t}}{\theta} - \frac{1}{2}\theta\left(\frac{1-e^{-\theta t}}{\theta}\right)^2\right))$$

Valuation of MBS

Non-Agency MBS value= $\sum_{t=1/12}^{T-1/12}[P_1(t)V_{1t} + P_2(t)V_{1t} + P_3(t)V_{2t}^{a}] + P_4(T)V_{3T}$ (19)

Agency MBS value= $\sum_{t=1/12}^{T-1/12}[P_1(t)V_{1t} + P_2(t)V_{1t} + P_3(t)V_{2t}^{b}] + P_4(T)V_{3T}$ (20)

Simulation

μ_r=0.04, θ=0.25 and σ_r=0.01.

The parameters of the approximation pricing formula of the MBS	
Initial House Price *S*	5,000,000
Initial Interest Rate r_0	0.02
Loan to Value(LTV)	0.8
Coupon rate r_c	0.04
Proportion of Investor ρ	0.2
Maturity *T*	30 (years)

Results

Table 3. MBS value

Non-Agency MBS			Agency MBS		
closed form	simulation	error	closed form	simulation	error
4154389.13	4151459.11	0.07%	4154277.41	4151347.59	0.07%

Results suggest that our approximation formula performs well because the error is smaller than 1 percent.

Loan to Value Effects

Non-Agency MBS			Agency MBS		
closed form	simulation	error	closed form	simulation	error
1556079.14	1555042.75	0.07%	1556079.14	1555042.75	0.07%
1815424.78	1814215.67	0.07%	1815424.78	1814215.67	0.07%
2074769.16	2073387.33	0.07%	2074769.16	2073387.33	0.07%
2334112.29	2332557.77	0.07%	2334112.29	2332557.77	0.07%
2593456.69	2591729.51	0.07%	2593456.69	2591729.51	0.07%
2852811.56	2850911.56	0.07%	2852811.56	2850911.56	0.07%
3112202.85	3110129.05	0.07%	3112202.85	3110129.05	0.07%
3371698.07	3369447.01	0.07%	3371698.06	3369447.00	0.07%
3631469.32	3629031.22	0.07%	3631468.97	3629030.86	0.07%
3891961.43	3889310.46	0.07%	3891954.23	3889303.27	0.07%
4154389.13	4151459.11	0.07%	4154277.41	4151347.59	0.07%

Proportion of Investor Effects

	Non-Agency MBS			Agency MBS		
rho	closed form	simulation	error	closed form	simulation	error
0.1	4156121.93	4153132.57	0.07%	4156010.22	4153021.05	0.07%
.11	4155948.65	4152965.22	0.07%	4155836.94	4152853.70	0.07%
.12	4155775.37	4152797.88	0.07%	4155663.65	4152686.36	0.07%
.13	4155602.09	4152630.53	0.07%	4155490.37	4152519.01	0.07%
.14	4155428.81	4152463.19	0.07%	4155317.09	4152351.66	0.07%
.15	4155255.53	4152295.84	0.07%	4155143.81	4152184.32	0.07%
.16	4155082.25	4152128.49	0.07%	4154970.53	4152016.97	0.07%
.17	4154908.97	4151961.15	0.07%	4154797.25	4151849.63	0.07%
.18	4154735.69	4151793.80	0.07%	4154623.97	4151682.28	0.07%
.19	4154562.41	4151626.46	0.07%	4154450.69	4151514.94	0.07%
0.2	4154389.13	4151459.11	0.07%	4154277.41	4151347.59	0.07%

Interest Rate Volatility Effects

	Non-Agency MBS			Agency MBS		
sigmar	closed form	simulation	error	closed form	simulation	error
0.1%	4169693.28	4167155.42	0.06%	4169605.69	4167067.95	0.06%
0.2%	4169212.22	4166626.32	0.06%	4169119.47	4166533.70	0.06%
0.3%	4168296.22	4165670.18	0.06%	4168198.48	4165572.59	0.06%
0.4%	4166945.28	4164287.00	0.06%	4166842.74	4164184.62	0.06%
0.5%	4165216.84	4162537.29	0.06%	4165110.41	4162431.04	0.06%
0.6%	4163213.48	4160523.98	0.06%	4163104.29	4160414.97	0.06%
0.7%	4161044.64	4158353.12	0.06%	4160933.71	4158242.38	0.06%
0.8%	4158805.06	4156102.92	0.07%	4158693.25	4155991.30	0.07%
0.9%	4156568.52	4153806.95	0.07%	4156456.50	4153695.13	0.07%
1.0%	4154389.13	4151459.11	0.07%	4154277.41	4151347.59	0.07%
1.1%	4152304.97	4149028.94	0.08%	4152193.94	4148918.12	0.08%
1.2%	4150341.92	4146480.90	0.09%	4150231.87	4146371.07	0.09%
1.3%	4148516.83	4143784.47	0.11%	4148407.97	4143675.85	0.11%
1.4%	4146839.95	4140917.32	0.14%	4146732.44	4140810.06	0.14%
1.5%	4145316.83	4137871.55	0.18%	4145210.78	4137765.78	0.18%

House Price Volatility Effects

	Non-Agency MBS			Agency MBS		
sigmah	**closed form**	**simulation**	**error**	**closed form**	**simulation**	**error**
1%	4150612.36	4147815.23	0.07%	4150612.36	4147815.23	0.07%
2%	4151015.77	4148204.05	0.07%	4151015.77	4148204.05	0.07%
3%	4151743.15	4148904.86	0.07%	4151743.15	4148904.86	0.07%
4%	4152832.68	4149954.78	0.07%	4152831.94	4149954.04	0.07%
5%	4154389.13	4151459.11	0.07%	4154277.41	4151347.59	0.07%
6%	4157696.10	4154702.77	0.07%	4155969.74	4152979.22	0.07%
7%	4166680.41	4163615.19	0.07%	4157608.15	4154556.25	0.07%
8%	4185603.51	4182464.87	0.08%	4158811.19	4155706.68	0.07%
9%	4215782.36	4212580.48	0.08%	4159321.77	4156179.36	0.08%
10%	4255302.63	4252060.01	0.08%	4159073.29	4155908.94	0.08%

Conclusion

- The close form solution for MBS are derived and the difference between the results of the closed form formula and the simulation is small and generally less than 0.1 percent