**Chapter 27 Solutions**

1. Itˆo’s lemma is a useful result because it allows the computation of stochastic differentials of arbitrary functions having as an argument a stochastic process that itself is assumed to possess a stochastic differential. In this respect, Itˆo’s formula is as useful as the chain rule of ordinary calculus. Given an Itˆo stochastic process S(t,w) with respect to a given Wiener process Z(t,w), and letting Y (t,w) = u[t, S(t,w)] be a new process, then based on Itˆo’s lemma and Taylor’s theorem, we can obtain

dt × dt = 0, dZ× dZ = dt, dt × dZ = 0

dY = utdt + uSdS + uSS (dS)2

A stochastic process is an Itˆo process if the random variable dS(t,w) can be represented by

dS(t,w) = μ[t, S(t,w)]dt + σ[t, S(t,w)]dZ(t,w)

where The first term, μ[t, S(t,w)]dt, is the expected change in S(t,w) at time t. The second term, σ[t, S(t,w)]dZ(t,w), reflects the uncertain term. dZ(t,w), is called white noise followed standard normal distribution; it denotes an infinitesimal change in the Wiener process, models financial uncertainty in continuous time.

1. The stock price has a constant expected return μ and the volatility of its return is constant σ times the Wiener process dZ(t,w) which follows normal distribution with zero mean and variance, t. Therefore, by equation (27.12) in section 27.3, the stock price follows a lognormal distribution with mean (μ- σ2)t and variance σ2t.
2. Assume the stock price follows the stochastic process as follows

dS(t,w) = μS(t,w)dt + σS(t,w)dZ(t,w)

The expected value of a call option is where k is the strike price in call option contract.

By equation (27.12), we can arrange the call option formula as

(1)

Where z is the normal distribution N(0, T).

To derive the Black–Scholes call option model, first we should prove equal to risk-free rater, r, in risk-neutral framework.

That is, under the risk-neutral measure Q.

Proof:

Therefore, and replace r into equation (1), we can obtain equation (2) as follows:

(2)

Let standard normal distribution Y, then we can rearrange equation (2) as

(3)

Therefore, equation (2) holds if and only if equation (3) holds. Then equation (1) can be written as

1. let and use Itˆo’s lemma to find

1. The stock price follows lognormal distribution where Z follows normal distribution with zero mean and variance, T.
   1. Given the information and in the first two years, then the stock price at the end of two years is

Where Z follows normal distribution with mean zero and variance, 2. Therefore, the stock price follows lognormal distribution with mean 4.172 (ln50+0.26 = 4.172) and variance 0.8 (0.22(2) = 0.8).

* 1. At the end of three years, given the information in the last year, the stock price can be written in terms of the stock price at the end of two years as

Where Z’ is the normal distribution with mean zero and variance, 1. From the part (a), we can obtain

Since Z and Z’ are two independent normal distributions, therefore the sum of these two independent normal distribution Z”= is following a normal distribution with mean zero and variance 0.2025 (0.22(2)+0.352 = 0.2025).

Then the stock price at the end of three years follows lognormal distribution with mean 4.6095 and variance 0.2025 ().

1. The stock price change following the stochastic process in question 2 is more appropriate than the stochastic process in question 6 because the parameters and are meaningful for the return of stock price and can be estimated by the expected return and standard deviation of the stock’s returns, respectively. In addition, investors prefer to know how much percent of return for their investment rather than the change of the investment.

1. When the stock price follows the stochastic process as follows:

dS(t,w) = μdt + σdZ(t,w), μ=1.5, σ=2 and S0 = $110

1. Then St follows a normal distribution N(S0+ μt, σ2t). Therefore, for next year, S1 ~ N(110+1.5, 22) = N(111.5, 4). It implies that the stock price in the next year follows a normal distribution with mean 111.5 and standard deviation 4.
2. 95% confidence limits for the stock price in the next year are between the meanstandard deviation, that is, [ 111.5- 4, 111.5+ 4] =[107.5, 115.5]
3. When the stock price follows the stochastic process as follows:

dS(t,w) = μ S(t,w) dt + σ S(t,w) dZ(t,w), μ=0.7, σ2=25% and S0 = $100

Then, the stock price follows lognormal distribution where Z follows normal distribution with zero mean and variance, T

1. At the end of six months,

Where Z1 follows normal distribution with zero mean and variance 0.5.

Therefore, the stock price follows lognormal distribution with mean 4.893 (ln100+ = 4.893) and variance 0.125 (0.52(0.5) = 0.125)

The expected stock price at the end of six months is e 4.893=133.353

1. The standard deviation of the stock price at the end of six months

is e 0.125=1.133

1. In next year,

Where Z2 follows normal distribution with zero mean and variance 1. Therefore, the expected stock price is

1. Assume the stock price follows the stochastic process as follows:

dS(t,w) = μ S(t,w) dt + σ S(t,w) dZ(t,w)

1. Y=3 S(t,w), then dY=3 dS(t,w). Therefore, Y follows the stochastic process as dY=3dS(t,w) = 3μ S(t,w) dt + 3σ S(t,w) dZ(t,w)
2. Y= Sn(t,w), then

dY= nSn-1(t,w) dS(t,w)+ n(n-1)Sn-2(t,w) [dS(t,w)]2

= nSn-1(t,w) [μ S(t,w) dt + σ S(t,w) dZ(t,w)]+ n(n-1)Sn-2(t,w) (σ2 ) S2(t,w)dt

= (nμ+ n(n-1) σ2 )Sn(t,w) dt + (n σ) Sn(t,w) dZ(t,w)

1. Y=e r(T-t) Sn(t,w), then

dY= -e r(T-t) Sn(t,w)dt+ e r(T-t) {nSn-1(t,w) dS(t,w)+ n(n-1)Sn-2(t,w) [dS(t,w)]2}

=(nμ+ n(n-1) σ2 -1) e r(T-t) Sn(t,w) dt + e r(T-t) (n σ) Sn(t,w) dZ(t,w)

1. The stock price follows lognormal distribution where Z follows normal distribution with zero mean and variance, T. The stock price in two years in equation 5 is

Therefore, the stock price follows lognormal distribution with mean 4.172 (ln50+0.26 = 4.172) and variance 0.8 (0.22(2) = 0.8).

The probability that stock price larger than $130 in two year can be calculated as

P()=P()

==P(X>7.7761)=0

Where X is the standard normal distribution and Z is the normal distribution with zero mean and variance, 2.