Credit Analysis of Corporate Credit Portfolios---
A Cash Flow Based Conditional Independent Default Approach

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Abstract

This study suggests combining a cash flow based structural credit model and a conditional independent default approach, the factor copula method, to estimate the multi-period credit risk of a corporate credit portfolio. The approach differs from most existing portfolio credit models in that it considers the risk dynamics and can endogenously estimate the recovery rate. The empirical results of applying it to the pricing of a market-traded CDX have shown that it performs well, especially for the model with a dynamic default threshold.

Keywords: Conditional independent default, Factor copula, Cash flow, Credit portfolio pricing, Risk dynamic

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Due to the implementation of New Basel Accord (or Basel II) and the fast development of collateralized debt obligations (CDO), portfolio credit analysis has become an important research area in recent years. Most existing studies are reduced from models and focus on handling the default correlation between component assets of a credit portfolio. To consider the issue, several approaches are developed in the literature such as conditional independent default approach (later denoted as CID approach), contagion models, and some other varieties.[1] Few of them incorporate the dynamics of risk structure and are able to endogenously estimate the portfolio recovery (loss) rate. Within the framework of structural form credit models and for analyzing the credit risk of a corporate credit portfolio, this research suggests a new approach combining a cash flow based credit model that has a factor structure and a conditional independent default method, the factor copula method.[2] Since the new approach is based upon a firm's future free cash flows, it can evade the controversies stemming from most traditional option-based structural models that employ a market-based valuation approach.[3]

The study employs a state-dependent free cash flow process to generate each component

[1] The major conditionally independent defaults (CID) studies include Duffee (1999), Zhou (2001), Schonbucher (2003), Driessen (2005), Bakshi, Madam and Zhang (2004), Janosi, Jarrow and Yildirim (2002), and Zhang (2003). The contagion models include the infectious default model (Davis and Lo, 1999) and the propensity model (Jarrow and Yu, 2001). The other varieties are for example, Giesecke and Weber (2004) and Hull and White (2006).

[2] Because a firm's free cash flow is mainly affected by both the firm's management policies and macroeconomic economic cycle, the free cash flow dynamics include both systematic factors and a firm specific effect.

[3] Most structural credit models are under Merton's (1974) framework. They assume that the stock return is normal-distributed. However, the literature has shown that stock return distribution is asymmetric, fat tailed, and volatility smiled. In addition, they also assume that an efficient market exists and a firm's value is not affected by its capital structure.

firm's multi-period asset value distributions and, therefore, its multi-period default probabilities (later denoted as PD) and recovery rates (later denoted as RR) endogenously[4] when the default threshold has determined. Multi-period portfolio loss distributions can then be obtained through conditional default approaches such as the factor copula method. The multi-period credit information is useful in the tranching and the pricing of credit portfolio by employing the method suggested by Geske (1977) and Jarrow and Turnbull (1995).

This study suggests a mean-reverting Gaussian process to model the common state factors underlying cash flow processes of the portfolio component firms, conforming to a common understanding that the growth rates of most economic indicators are weakly stationary. The estimated forward-looking state factor information is useful in each firm's cash flow simulation. Each component firm's multi-period unconditional asset value distributions can be spawned by its free cash flow process. With default boundary information, we are able to estimate each component firm's unconditional multi-period PDs and RRs endogenously and concurrently. Because all component firms' cash flow processes are affected by the same common state factors, their PDs are independent conditioning on a given state vector (or factor paths). Under this conditional independent setting, this research is able to obtain joint default probability density function of a credit portfolio.

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[4] Free cash flow to firm is a firm's operating free cash flow prior to the payment of interests to the debt holders and after deducting the funds required to maintain the firm's productivity (i.e. non-discretionary capital expenditures). Free cash flow to firm is a measure to estimate the value of the total firm. On the other hand, free cash flow to equity is used to estimate the value of a firm's equity and is equal to the free cash flow to firm minus debt repayments.

Yu (2005) showed empirical evidence of non-vanishing short term spreads of In addition, for Merton-type structural models, empirical evidence of non-vanishing short term spreads is presented in Yu (2005) and is interpreting the phenomenon as "transparency spreads" due to incomplete information[5]. Giesecke (2004) presented a structural model in which investors have incomplete information for either the firm's asset value, or the default threshold or for both.[6] In Giesecke's model, investors update their belief on the joint default threshold distribution whenever a component firm defaults or its asset value reaches a new historical low. However, Giesecke's model is unable to forecast future loss distribution because people cannot observe future defaults. This work follows the concept of Giesecke (2004) and relaxes the assumption of complete information in traditional structural credit models by a dynamic default threshold setting, considering changes in a firm leverage distribution and being able to address the credit spread underestimation issue of Merton type structural models. The integrated approach incorporates default correlation and default threshold dynamics in portfolio (such as CDO) credit analysis and valuation.

To demonstrate the application of the new approach, this research provides an empirical evidence of evaluating a market-traded CDX comprising thirty corporate bonds under both

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[5] Some studies relax the complete information assumption to solve the predictable issue of Merton type models. Duffie and Lando (2001) assume that investor have imperfect information on firm values due to periodic and imperfect accounting reports. Giesecke and Goldberg (2004) suggest a random and unobservable default threshold.

[6] Giesecke (2004) adopts a Bayesian analysis to model default threshold dynamics and formulates a default threshold copula. The model is not predictable and is able to solve the issue of non-vanishing short-term credit spread of Merton type credit models.

constant and dynamic default threshold settings. Results show that the proposed model performs well, especially when combining with a dynamic default threshold.

The rest of the paper is divided into four sections: Section I presents the settings of the cash flow-based credit portfolio model with a constant and a dynamic default threshold. Section II implements the proposed approach in pricing a market-traded CDX. Section III discusses some further extensions of the proposed model. Section IV concludes this study.

I. The Model

This section introduces a firm's cash flow model developed by Liao, Chen, and Lu (2006) as the foundation for single firm credit risk assessment and, based on the state-dependent characteristics of the cash flow model, incorporates the factor copula method to extend the single-firm model to a portfolio credit model. This section also discusses the credit risk models by settings of constant and dynamic default thresholds.

A. A firm's state-dependent cash flow credit model

According to Liao, Chen, and Lu (2006), a firm's cash flow is mainly determined by its long-term average level, systematic state shocks and firm specific shocks. They establish the relationship between the i^{th} firm's C_{it} and the state of the economy as Eq. (1). In Eq. (1) the i^{th} firm's C_{it} is affected by both a set of k systematic factors and an idiosyncratic (firm specific) effect. In addition, F_{jt} indicates the unobservable state factors; α_{ij} indicates the sensitivities of the i^{th} firm's C_{it} to the j^{th} state factor; and ξ_{it} indicates the i^{th} firm's idiosyncratic factor representing the part the variations of the i^{th} firm's C_{it} that can not be explained by the state factors and is normally distributed with mean zero and variance equal to residual variance not explained by the systematic factors, that is $1-h_i$ where h_i indicates the variance explained by the systematic factors. According to Liao, Chen, and Lu (2006), in most cases, a firm's free cash flow to firm follows a mean-reverting (weakly stationary) process, the number of factors (k) and the factor loading α_{ij} can therefore be estimated by factor analysis that extracts the unobservable common factors underlying the free cash flows of the component firms of a credit portfolio.

$$C_{it} = E(C_{it}) + \sum_{j=1}^{k} \alpha_{ij} F_{jt} + \xi_{it} \qquad \xi_{it} \sim N(0, \sqrt{1-h}_{it}) \tag{1}$$

To take into consideration of the changes in risk structure, the model employs a mean-reverting Gaussian process to describe each state factor process as Eq. (2).

$$dF_{jt} = a_{F_j}[b_{F_j} - F_{j,t-1}]dt + \sigma_{F_j} dz_j \tag{2}$$

Where, F_{jt} indicates the j^{th} state factor value in the time t; a_{F_j} indicates the mean-reverting speed of F_{jt}; b_{F_j} is the long-term average level of F_{jt}; σ_{F_j} indicates the standard deviation of the term variation of F_{jt}, and dz_j is a wiener process. Assuming that the stochastic characteristics of the economy will not structurally change in foreseeable future, the parameters of each state factor's process are set constant. Combining Eq. (1) and Eq. (2), many probable free cash flow paths can be simulated and corresponding firm value paths can be obtained by Eq. (3).

$$V_{it} = \left[\sum_{\tau=t+1}^{T} \frac{C_{i\tau}}{(1+\gamma_A)^{\tau-t}} \right] + \frac{C_{iT}(1+g)}{(1+\gamma_A)^{T-t}(\gamma_A - g)} \tag{3}$$

When the default boundary ($\overline{L}_t$) is given, the probability of default for the i^{th} firm at time t (denoted as PD_{it}) is defined as Eq. (4). When a default occurs, the recovery rate at time t (later denoted as RR_{it}) can be written as Eq. (5). The current model endogenously determined the two main credit risk indicators, PD_t and RR_t. In Eq. (4), $f_{it}(V)$ indicates the unconditional distribution of the i^{th} firm's asset value at time t.

$$PD_{it} = \int_{-\infty}^{\overline{L}_t} f_{it}(V)dV \tag{4}$$

$$RR_{it} = \frac{1}{\overline{L}_{it}(PD_{it})} \int_{0}^{\overline{L}_{it}} V f_{it}(V) \cdot dV \,. \tag{5}$$

B. The construction of portfolio credit risk model by factor copula

This study employs the conditional independent default approach (the CID approach) to handle the default correlation and to some extent the default contagion between and among the component firms in a credit portfolio. Due to the factor model setting of the firm's free cash flow, this research uses factor copula method, one of the CID approaches, to extend previous single-firm credit model to a portfolio credit model.

Since the single firm cash flow model in Eq. (1) is set as a factor model, indicating that a firm's free cash flow to firm is influenced by a set of systematic factors and a firm specific effect. Conditioning on the realization of a set of state vector path, the diffusion terms of component firms' cash flow process are independent. It implies that the firm value distributions of the component firms are independent given a specific state vector path. We can then obtain, at time *t*, the portfolio's conditional joint value distribution given a realized state vector path can be expressed as $\prod_{j=1}^{n} f_t^j\left(V_t^j \middle| \tilde{F} = \bar{F}\right)$, where $f_t^j\left(V_t^j \middle| \tilde{F} = \bar{F}\right)$ indicates the probability density function of the component firm *j's* value at time *t* given the *F* state vector path. Employing the factor copula method, the conditional cumulative joint value distribution of the credit portfolio for each future time point *t* is as in Eq. (6), where $f_t(\tilde{F})$ is the probability density function of the state vector $\tilde{F}$ at time *t*. The conditional joint probability density function of the portfolio component firm's value for each future time point *t* is as in Eq. (7). In discrete cases, when simulating a large number of state paths (*S* paths) by the state process in Eq. (2), the Eq. (7) can be approximated by Eq. (8) according to the Law of Large Number and the Central Limit Theorem. In Eq. (8), $P_t\left(F_i\right)$ indicates the probability of state vector path F_i at time *t* and is equal to $1/S$. With the conditional joint probability density function of the credit portfolio component firm's value at time *t*, we can obtain the portfolio's *PD* and *RR* at time t endogenously.

$$
\begin{aligned}
F^{P}(V_t^{i},\ldots,V_t^{n}|\tilde{F}) &= \int \prod_{i=1}^{n}\left(\int f_t\left(V_t^{i}\middle|\tilde{F}=\overline{F}\right)dV^{i}\right) f\left(\tilde{F}\right)d\tilde{F} \\
&= \int \prod_{i=1}^{n} F_i\left(V_t^{i}\middle|\tilde{F}=\overline{F}\right) f\left(\tilde{F}\right)d\tilde{F} \\
&= C\left(F_1\left(V_t^{1}\middle|\tilde{F}\right),\ldots,F_n\left(V_t^{n}\middle|\tilde{F}\right)\right)
\end{aligned}
\tag{6}
$$

$$
f^{P}\left(V_t^{1},\ldots,V_t^{n}\middle|\tilde{F}\right)=\frac{\partial^{n}F^{p}\left(V_t^{1},\ldots,V_t^{n}\middle|\tilde{F}\right)}{\partial(V_t^{1})\cdots\partial(V_t^{n})}=\int f\left(\tilde{F}\right)d\tilde{F}\cdot\prod_{i} f_i\left(V_t^{i}\middle|\tilde{F}\right)
\tag{7}
$$

$$
\begin{aligned}
f_t^{P}\left(V_t^{1},\ldots,V_t^{n}\middle|\tilde{F}\right) &\approx \sum_{i=1}^{S}\prod_{j=1}^{n} f_t^{ij}\left(V_t^{ij}\middle|\tilde{F}=F_i\right)P_t\left(F_i\right) \\
&\approx \frac{1}{S}\sum_{i=1}^{S}\prod_{j=1}^{n} f_t^{ij}\left(V_t^{ij}\middle|\tilde{F}=F_i\right)
\end{aligned}
\tag{8}
$$

When assessing a firm's credit risk, we need to determine the value of the default boundary L_i for each firm. A firm defaults when its value falls below a certain default boundary.

C. Constant and Dynamic default threshold settings

Giesecke (2004) states that, in the real world, investors have imperfect information about a firm's default threshold and form default threshold distribution belief based on realized asset values. Accordingly, this study relaxes the assumption of fixed default threshold. Default threshold dynamics are set based on the cash flow information derived by conditional independent default approach due to the assumption that cash flow correlation is the major source of default correlation. Two different settings for default threshold dynamics are illustrated as follows.

C.1. Constant Default Threshold

The constant threshold (L_t) is the sum of current liability and half of the long-term liability according to KMV model. This setting serves as the base case to reveal the impact of dynamic default threshold dynamics on the portfolio expected loss (later denoted as EL).

The first passage default approach defines that a default event occurs when asset value hits default threshold for the first time and defines the time to default as the first passage time $\hat{\tau}$. Given a firm defaults ($L > V_t$ for the first time) at the first passage time $\hat{\tau}$, we obtain the following firm loss rates (FLR):

$$FLR_t = \begin{cases} 0, & if\ \hat{\tau} > t \\ 1 - \dfrac{V_t}{L}, & if\ \hat{\tau} = t \\ 1, & if\ \hat{\tau} < t \end{cases} \tag{9}$$

C.2. Default Threshold with Stationary Leverage Ratio

Relaxing investors' perfect information assumption, this work incorporates default threshold dynamics into the previous setting. Investors formulate their default threshold (L_t) assessment based on asset value realized in previous periods. Giesecke (2004) assumes in his model that investors gather more information on default threshold when one firm reaches an historical low of asset value. If the firm survives, investors infer the default threshold must be lower than historical low of asset value. This study follows this logic, setting $\hat{L}_t$, the upper bound of default threshold range, as historical low of simulated asset value.

This setting also incorporates a firm's historical leverage ratio information by assuming that investors believe firms maintain their leverage ratios at a stationary level (ℓ) according to the proposition of stationary leverage ratio suggested by Collin-Dufresne and Goldstein (2001). Collin-Dufresne and Goldstein suggest that firms adjust their capital structure to reflect changes in asset value. Their argument is supported by the market credit spread data. Investors believe that firms adjust their debt level (L_t) according to the asset value realized in the previous period V_{t-1} with the attempt to maintain a stationary leverage ratio. Based on this idea, firms most likely adjust their debt level until the ratio of current debt level to the latest realized asset value (L_t/V_{t-1}) equals to long-term average leverage ratio (ℓ). For simplification, we suggest a triangular shaped distribution with probability density peaks at long-term average leverage ratio. Equation (10) and (11) describe the probability density function and can be graphed as figure 1.

[Insert Figure 1 here]

If $0 < \ell V_{t-1} < \hat{L}_t$:

$$f(x) = \begin{cases} \dfrac{2}{\ell V_{t-1}\hat{L}_t} x, & if \quad 0 < V_t < \ell V_{t-1} \\ \dfrac{2}{\hat{L}_t\left(\hat{L}_t - \ell V_{t-1}\right)}\left(\hat{L}_t - x\right), & if \quad \ell V_{t-1} < V_t < \hat{L}_t \\ 0, & if \quad V_t > \hat{L}_t \end{cases} \tag{10}$$

If $\ell V_{t-1} > \hat{L}_t$:

$$f(x)=\begin{cases}\dfrac{2}{\hat{L}_t^{\,2}}x, & if\ 0<V_t<\hat{L}_t\\ 0, & if\ V_t>\hat{L}_t\end{cases} \tag{11}$$

Given a realization of asset value path, a firm's marginal probability of default during period t (PD_t) is the area between the range $(V_t, \hat{L}_t)$ of probability density curve in figure 1 when $V_t < \hat{L}_t$ and equals zero when $V_t > \hat{L}_t$.

If $0 < \ell V_{t-1} < \hat{L}_t$:

$$PD_t = \mathrm{P}(V_t < L_t) = \begin{cases} 1-\dfrac{V_t^2}{\hat{L}_t(\ell V_{t-1})}, & if\ 0<V_t<\ell V_{t-1}\\ \dfrac{(\hat{L}_t - V_t)^2}{\hat{L}_t(\hat{L}_t-\ell V_{t-1})}, & if\ \ell V_{t-1}<V_t<\hat{L}_t\\ 0, & if\ V_t>\hat{L}_t\end{cases} \tag{12}$$

If $\ell V_{t-1} > \hat{L}_t$:

$$PD_t = \mathrm{P}(V_t < L_t) = \begin{cases} 1-\left(\dfrac{V_t}{\hat{L}_t}\right)^2, & if\ 0<V_t<\hat{L}_t\\ 0, & if\ V_t>\hat{L}_t\end{cases} \tag{13}$$

The cumulative survival rate (SR_t) which represents the probability one firm survives to the end of period t, is shown as Eq. (14). The cumulative probability of default (Ω_t) is expressed as Eq. (15).

$$SR_t = \prod_{\tau=1}^{t}(1-PD_\tau) \tag{14}$$

$$\Omega_t = 1 - SR_t \tag{15}$$

If $V_t < \hat{L}_t$ and $0 < \ell V_{t-1} < \hat{L}_t$, the marginal expected recovery rate (Λ_t) is derived as Eq. (16).

$$\Lambda_t = (1 - PD_t) + PD_t E(\frac{V_t}{L_t} \mid L_t > V_t) \tag{16}$$

$$= \begin{cases} \Lambda'_t = \dfrac{V_t^2 + 2V_t(\ell V_{t-1} - V_t)}{\hat{L}_t \cdot \ell V_{t-1}} + \dfrac{2V_t}{\hat{L}_t(\hat{L}_t - \ell V_{t-1})}\left(\hat{L}_t \ln \dfrac{\hat{L}_t}{\ell V_{t-1}} - \hat{L}_t + \ell V_{t-1}\right), & if\ 0 < V_t < \ell V_{t-1} \\ \Lambda''_t = 1 - \dfrac{\left(\hat{L}_t - V_t\right)^2}{\hat{L}_t\left(\hat{L}_t - \ell V_{t-1}\right)} + \dfrac{2V_t}{\hat{L}_t\left(\hat{L}_t - \ell V_{t-1}\right)}\left(\hat{L}_t \ln \dfrac{\hat{L}_t}{V_t} - \hat{L}_t + V_t\right), & if\ \ell V_{t-1} < V_t < \hat{L}_t \end{cases}$$

If $V_t < \hat{L}_t$ and $\ell V_{t-1} > \hat{L}_t$, the marginal expected recovery rate is as Eq. (17).

$$\begin{aligned} \Lambda_t &= (1 - PD_t) + PD_t E(\frac{V_t}{L_t} \mid L_t > V_t) \\ &= \left(\frac{V_t}{\hat{L}_t}\right)^2 + \frac{2V_t\left(\hat{L}_t - V_t\right)}{\hat{L}_t^{\,2}} \end{aligned} \tag{17}$$

With the formulae above, the FLR formula is as Eq. (18):

$$FLR_t = \begin{cases} 1 - SR_{t-1}\Lambda'_t, & if\ 0 < V_t < \ell V_{t-1} < \hat{L}_t \\ 1 - SR_{t-1}\Lambda''_t, & if\ \ell V_{t-1} < V_t < \hat{L}_t \\ 1 - SR_{t-1}\Lambda_t, & if\ 0 < V_t < \hat{L}_t < \ell V_{t-1} \\ 1 - SR_{t-1}, & if\ V_t > \hat{L}_t \end{cases} \tag{18}$$

The above threshold setting is the extreme concerning how much effect historical long-term average leverage ratio (ℓ) has on the distribution of current unobservable default threshold. In future extension, a trapezoid distribution with a level probability density around ℓV_{t-1} is a good substitute. A trapezoid distribution reflects the belief that firms maintain their leverage ratios around their long-term average.

With the simulated results of PD and FLR conditionally on a state vector path, portfolio loss rates (later denoted as PLR) can be estimated by the factor copula method. By generating many state vector paths, the multi-period portfolio loss distributions can be determined and the multi-period loss distributions can be derived for evaluating credit portfolio.

II. Empirical Applications

This section illustrates the model effectiveness by providing empirical evidences of credit portfolio pricing by the proposed model. The credit portfolio for the empirical analysis is the marketable Credit Default Swap index (later denoted as CDX) including thirty component firms. The following introduces selection of cash flow proxy, the CDX data, parameter estimation of the cash flow model, and some other required parameters such as growth rate and weighted average cost of capital. This section also illustrates adjustments when implementing the model and compares the model results with the real market quote of the CDX.

A. The Selection of Cash Flow Proxy

Free cash flow to "firm" (later denoted as *FCFF*), instead of that to "equity", is used to estimate a firm's asset value distribution, which is the major foundation for performing structural form credit models The *FCFF* is defined as Eq. (19). In Eq. (19), C_t^O denotes a firm's operating cash flow; E_t^C denotes non-discretionary capital expenditure; E_t^{ℓ} denotes expenditures for capital leases; F_t^{C} denotes an increase in funds for construction, and ppe_t^{γ} denotes reclassification of inventory to property, plant, and equipment.

$$FCFF_t = C_t^o - E_t^c,\ E_t^C = E_t^{\ell} + F_t^c + ppe_t^r \tag{19}$$

B. The CDX data

The underlying credit portfolio is the Dow Jones 5-year NA.IG.HVOL (it is the abbreviation of North American, investment grade, high volatility) CDX that comprises thirty straight unsecured senior corporate bonds of 30 different U.S. listed firms in Markit website. Its quote of Series 7 is effective on Sep 21, 2006. The underlying credit portfolio selection criteria are as follows: First, the CDX must be composed of non-financial firms. Second, CDX with many component firms lacking financial data is excluded. The Dow Jones 5-year NA.IG.HVOL CDX meets the above criteria. The related data, including the historical data of CDX and CDS, are acquired from the Markit database and the Lighthouse Database of Bank of America.

For the component firms of the NA.IG.HVOL CDX, three firms are excluded due to the lack of sufficient financial information[7]. Their respective credit information of multi-period expected loss rates are represented by the Moody's idealized expected loss table. In addition, two firms of other 27 component firms, have some missing financial data in one

[7] For the three excluded firms, they (RESCAP and CXR) are unavailable for the financial data and one (CTX) has a strongly unstable historical free cash flow pattern that doesn't fit our cash flow model so good.

period. The insufficient free cash flow information was completed by using their respective industry's average free cash flow adjusted by the scale effect. Twenty-seven component firms left and all of them are non-financial firms. We set our pricing date at December 29, 2006. All firm-related financial information and credit rating information is obtained from COMPUSTAT database and Bloomberg. The component firms' information of the CDX is illustrated in Table 1. In addition, this 5-year CDX spread ranges from 64 basis points to 98.5 basis points during 2006. The proposed model pricing spreads compare with this range to measure the model effectiveness.

[Insert Table 1 here]

C. The Factor Analysis and Parameter Estimation of the Stochastic State Model

According to the cash flow model set as equation Eq. (1) and Eq. (2), the factor analysis is used to obtain the relationship between a firm's cash flow and state factors. Factor analysis technique not only extracts state factors but provides the relationship such as Eq. (1). The twenty-seven firms' quarterly moving-average free cash flows per unit asset are viewed as the input data for factor analysis and the estimation period for the parameters of the proposed model is from 1999 Q1 to 2006 Q4. Seven factors are extracted with eigenvalues greater than unity and explain about 85.31% of these firms' free cash flow variation. Simultaneously, Factor analysis generates factor loadings on each firm's free cash flows and the time-series state factor values. The time-series state factor values are inputs to maximum likelihood estimation

(MLE) method to estimate the parameters of the stochastic state model. The estimated parameters of stochastic state model are illustrated in Table 2. With the estimates of these parameters, we simulate 100,000 paths for each factor, serving as the foundation of conditional independent default assessment. Additionally, the corresponding firm-specific risk factor (ξ_{it})[8] value to the 100,000 paths of the state factors are also generated. Based on each set of simulated factor paths and each firm's factor loadings, Eq. (1) can be utilized to derive the corresponding 100,000 cash flow paths of each firm.

[Insert Table 2 here]

D. Parameters Estimations of the Present Value Model

Based on the present value model shown in Eq. (3), multi-period firm value distributions formed by the 100,000 cash flow paths can be obtained. The parameters of present value model, the weighted average cost of capital (later denoted as WACC) and the constant growth rate, have to be estimated firstly. In addition, a cash flow mean shift term has to be estimated , which reconciles a firm's present value with its ordinary market value

D. 1. Estimation of a firm's weighted average cost of capital

A firm's WACC (γ_A) is composed of equity required return (γ_e) and cost of debt (γ_d). A one-factor CAPM estimates a firm's equity required return. The needed parameters are: the risk free rate, the market risk premium, and a company's market beta. The market risk

[8] We generate the random term with standard normal random variables multiplied by a scale factor equal to square root of the specific variance for each firm obtained from factor analysis.

premium of the U.S. market is set as 7.5% according to *Ibbostson Associates, annual*.[9] The market beta of each sample company is obtained from the COMPUTSTAT and the risk free rate proxy is the market rate of 10-year U.S. treasury notes obtained from Federal Reserve Bank of St. Louis. For simplicity, the cost of debt is assumed constant and is the market rate of the corporate bonds that have the same credit rating. With all the information, WACC for each component firm can be estimated. The estimated WACC is illustrated in Table 3.

[Insert Table 3 here]

D. 2. Estimation of constant growth rate of a firm

This study assume a firm grows at a constant rate after 10 years from the pricing time, that is, the beginning time of constant growth, T, in Eq. (3) is set as 10 years. For simplicity, the average of 10-year U.S. GDP growth rate 1.2975% (quarterly) is proxy for firms' constant growth rate. The GDP growth rate information can be obtained from Federal Reserve Bank of St. Louis.

D. 3. Estimation of the shift term to a firm's cash flow paths

Originally, market asset value was used to estimate the implied constant growth rate for each company. However, some companies' implied constant growth rates cannot be obtained, because of their negative future free cash flows. To reconcile the present value of future free cash flows with the current market asset value,[10]this study calibrate each component firm's

[9] Source: *Stocks, Bonds, Bills and Inflation* (Chicago, III.: *Ibbostson Associates, annual*).
[10] The current market asset value is transformed from equity market value according to the Merton (1974) model.

implied shift term (m) in cash flow by optimization technique and Eq. (20). With the implied shift term, we then add implied shift term back to the firm's each cash flow path, asset value in each future period can be calculated. The results are shown in Table 4.

$$V_{i0} = \left[\sum_{\tau=1}^{T} \frac{C_{i\tau} + m}{(1+\gamma_A)^{\tau}} \right] + \frac{(C_{iT} + m)(1+g)}{(1+\gamma_A)^{T}(\gamma_A - g)} \tag{20}$$

The purpose of the shift term is to make the simulated cash flows more realistic. In most cases, the shift term helps to increase the weight of the cash flows in the first T periods, which is more consistent with our original assumption. After the adjustment, the cash flows in the first T periods count about 40% for the present asset value on average. Note that although the asset market value implied shift term changes the mean of the cash flows, it does not change the volatility of them. That is, while getting a more realistic cash flow mean, the correlation structure still maintain and also the default correlation.

[Insert Table 4 here]

E. The Settings of Default Thresholds

This study separately implements the empirical works under a constant and a dynamic default threshold setting. The default threshold settings are discussed in subsection C of section I.

E. 1. Constant Default Threshold

Based on the previous discussions in subsection C.1 of section I, a firm's default threshold is the sum of its current liability and a half of long-term liability and a firm's expected loss rate is shown as Eq. (9).

E. 2. Dynamic Default Threshold with Stationary Leverage Ratio

Based on the previous setting in subsection C.2 of section I, a firm's default threshold process is only affected by its asset value path but uncorrelated with other component firms when a state vector path is given. The ratio of the debt book value to the sum of debt book value and equity market value are the proxy for leverage ratio. The average of the leverage ratios from 1997 to 2006 estimates its long-term average. A firm leverage ratio is mean-reverting around the long-term average.

F. Credit Analysis of the NA.IG.HVOL CDX

The above steps enable us to construct the multi-period loss distribution for each component firm. We sum up each firm's weighted loss rate of each state at each time point to get the multi-period portfolio loss distribution. It should be noted that when calculating the portfolio's loss rate, the three previously excluded component firms must be considered.

To assess the 5-year NA.IG.HVOL CDX, its 5-year expected portfolio loss rate needs estimation. Taking the weighted average of the 27 component firms' expected loss rate (FLR) given a state vector path derives the conditional portfolio expected loss rate (PLR). As for the rest three component firms (CTX, RESCAP, and CXR), we employ two different measures for estimating their expected loss rates, both market-based and historical data. For the market-based data, the average credit spreads of market-traded CDS with credit ratings corresponding to the three component firms are as our proxies respectively. For the historical data, Moody's idealized expected loss rates table (1982-2007) with credit ratings corresponding to the three component firms are as the proxies respectively. After simulating 100,000 state vector paths, we obtain the unconditional multi-period portfolio loss distribution. Figure 2 and figure 3 illustrate the multi-period portfolio loss distributions with constant and dynamic default boundaries. Figure 4 and figure 5 illustrate the fifth-year portfolio loss distributions with constant and dynamic default boundaries. The above figures are illustrated under the scenario that the three firms' expected loss rates are the Moody's historical expected loss rates.

[Insert Figure 2 and 3 here]

[Insert Figure 4 and 5 here]

Since literature in loss distribution has observed that it has the characteristics in scale and shape of distributions such as Beta, Gamma, and Weibull, we do some distribution fittings for the obtained portfolio loss distribution. The results of the parameter estimates and their standard deviations listed in Table 5 show that the distribution fittings are efficient.

[Insert Table 5 here]

Based on the fifth-year portfolio loss rate distributions with the constant and dynamic default threshold settings shown in figure 4 and 5, we can reasonably assess the 5-year NA.IG.HVOL CDX. The real quotes of the 5-year NA.IG.HVOL CDX in 2006 are within the range of 64.0 bps and 98.5 bps. Additionally, its quotes range from 79.75 bps to 84..25

bps in December, 2006. For our model with constant default threshold setting, the results show that the calculated credit spread is separately 36.39 bps and 41.37 bps for the situations that the proxy of the rest three firms' expected loss rate is employed as the market-traded CDS and the Moody's idealized expected loss rate. For our model with dynamic default threshold setting, the results show that the calculated credit spread is separately 84.25 bps and 89.23 bps for the two above situations. These results are presented in Table 6.

[Insert Table 6 here]

Table 6 shows that the model with the dynamic default threshold setting performs well in pricing of the 5-year NA.IG.HVOL CDX and the other underestimates its credit spreads. This may be due to the fact that the underlying portfolio is composed of more corporate bonds with the firm's characteristics of high volatility. The dynamic default threshold setting can account for more investors' imperfect information on the threshold.

III. Further Extensions

Several extensions can improve the proposed model. First, the current model may omit some systematic factors. Incorporating some market-wide indicators, such as the fragile indicators, into the factor analysis of portfolio component firm free cash flows may fix the gap. Second, replace the current simple dynamic default threshold setting with another more complex form.

IV. Conclusions

Within the framework of structural form credit models, this study proposes an integrated approach, incorporating a cash flow based model and a conditional independent default approach, to estimate multi-period credit risk of a corporate credit portfolio endogenously. Additionally, this work also considers different default threshold settings, including a constant and a dynamic threshold setting. Empirical results of applying the proposed new approach to the NA.IG.HVOL CDX market data show an acceptable performance in default risk pricing, especially under the dynamic default threshold setting. For further extension, the model can be improved by adding macroeconomic factors to capture more state change information, and by designing a more complicated dynamic threshold setup.

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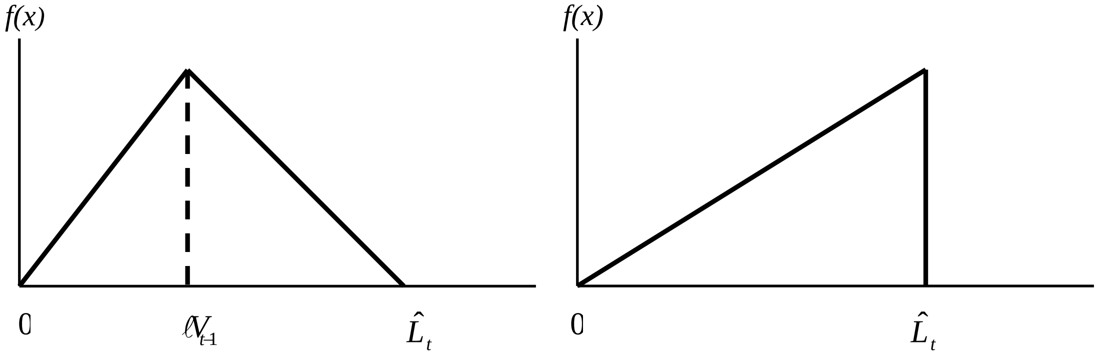
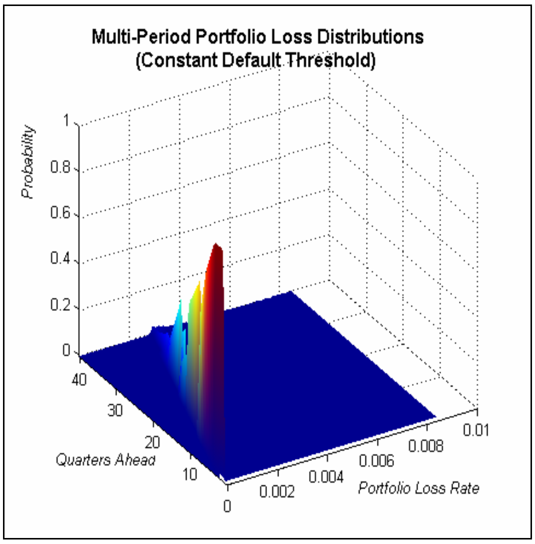


(A) When $0 < \ell V_{t-1} < \hat{L}_t$ (B) When $\ell V_{t-1} > \hat{L}_t$

This setting incorporates the information of leverage ratio. $\hat{L}_t$ is set as the upper bound of default threshold range, as the historical low of simulated asset value followed by Giesecke (2004). Furthermore, we assume that investors believe that firms adjust their debt level L_t according to the asset value realized in the previous period V_{t-1} with the attempt to maintain a stationary leverage ratio. Based on the idea, we assume that investors consider it most likely that firms adjust their debt level until current debt level divided by latest realized asset value (L_t / V_{t-1}) is equal to long-term average leverage ratio. When $0 < \ell V_{t-1} < \hat{L}_t$, we suggest a simplified triangular shaped distribution with probability density peaks at long-term average leverage ratio ℓ. The triangular shaped distribution is illustrated in (A). On the other hand, the probability density peaks at the upper bound $\hat{L}_t$ when $\ell V_{t-1} > \hat{L}_t$. The setting is illustrated in (B). The probability density functions for these two cases are given in equations (10) and (11) respectively.

Figure 1. Probability Density of Default Threshold



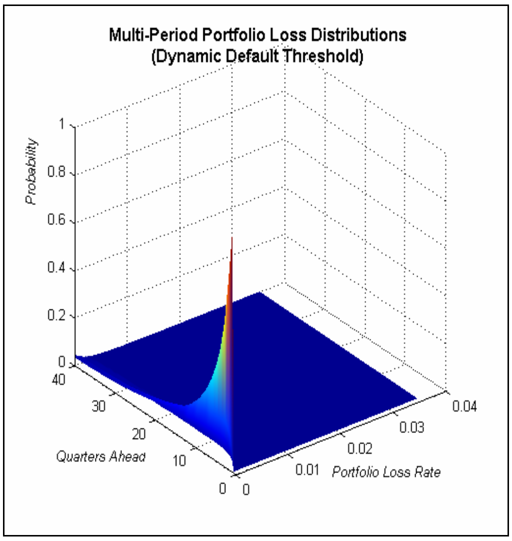
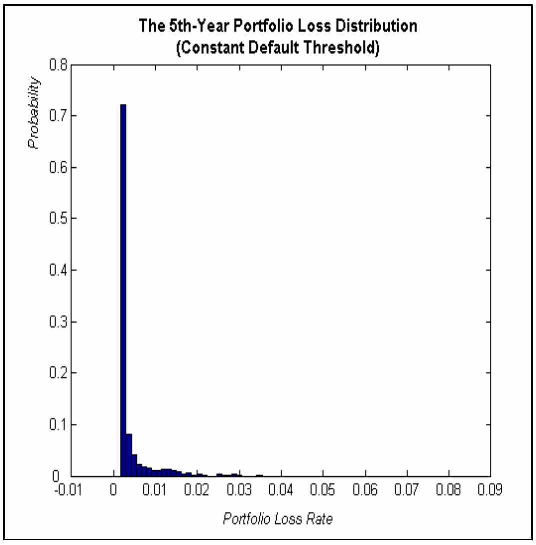


Figure 2. Multi-Period Portfolio Loss Dist. (Constant Default Threshold)

Figure 3. Multi-Period Portfolio Loss Dist. (Dynamic Default Threshold)

Figure 2 and 3 separately show the multi-period portfolio loss distributions with the constant and dynamic default threshold settings. This z axis indicates the probability corresponding to portfolio loss rate shown in the x axis. The y axis indicates the future points in time (quarterly). The integral of the distributions are the multi-period expected loss rate of the credit portfolio.



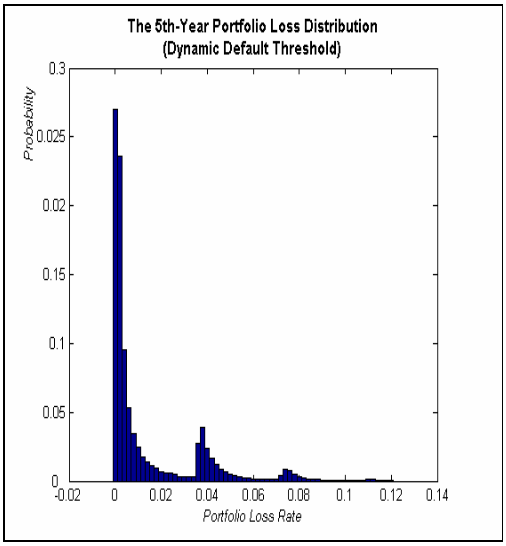


Figure 4. The 5th Y Portfolio Loss Dist. (Constant Default Threshold)

Figure 5. The 5th Y Portfolio Loss Dist. (Dynamic Default Threshold)

Figure 4 and 5 separately show the fifth-year portfolio loss distributions with the constant and dynamic default threshold settings. This vertical axis indicates the density corresponding to the portfolio loss rate shown in the horizontal axis.

Table 1. Characteristics of the Component Firms for the NA.IG.HVOL CDX

Table 1 shows the characteristics of the component firms for the NA.IG.HVOL CDX. They include the ticker, firm full name, its industry, the weight of the CDX portfolio, and its credit rating. The data sources contain Markit, COMPUSTAT, and Bloomberg databases.

Ticker	Firm	Industry	Weight	Rating
ARW	Arrow Electronics Inc	Electronic Parts	3.33%	Baa3
AZO	Autozone Inc	Auto And Home Supply Stores	3.33%	Baa1
CBS	CBS Corp	Television Broadcast Station	3.33%	Baa2
CCU	Clear Channel Communication	Broadcasting- Radio	3.33%	B1
CTL	CenturyTel Inc	Phone Comm Ex Radiotelephone	3.33%	Baa2
CTX	Centex Corp	Operative Builders	3.33%	Baa2
CXR	COX Communications Inc	Radio Broadcasting Stations	3.33%	Baa3
EQ	Embarq Corp	Phone Comm Ex Radiotelephone	3.33%	Baa3
EXPE	Expedia Inc	Transportation Services	3.33%	Ba1
GPS	Gap Inc	Family Clothing Stores	3.33%	Ba1
HET	Harrah's Operating Co Inc	Misc Amusement & Rec Service	3.33%	Ba2
HOT	Starwood Hotels & R Inc	Hotels, Motels,Tourist Courts	3.33%	Baa3
IACI	IAC/InterActiveCorp	Transportation Services	3.33%	Baa3
IP	International Paper Co	Paper And Allied Products	3.33%	Baa2
JNY	Jones Apparel Group Inc	Womens,Misses,Jrs Outerwear	3.33%	Baa3
LEN	Lennar Corp.	Operative Builders	3.33%	Baa2
LTD	Ltd Brands Inc	Women's Clothing Stores	3.33%	Baa2
MWV	MeadWestvaco Corp	Paperboard Mills	3.33%	Baa2
OLN	Olin Corp	Rolling & Draw Nonfer Metal	3.33%	Baa3
PHM	Pulte Homes Inc	Operative Builders	3.33%	Baa2
RESCAP	Residential Capital LLC	N.A.	3.33%	Baa3
RRD	RR Donnelley & Sons Co	Commercial Printing	3.33%	Baa1
RSH	RadioShack Corp	Radio,TV,Cons Electr Stores	3.33%	Ba2
SHW	Sherwin-Williams Co	Pants, Varnishes, Lacquers	3.33%	A3
SWY	Safeway Inc.	Grocery Stores	3.33%	Baa3
TIN	Temple-Inland Inc	Paperboard Mills	3.33%	Baa2
TOL	Toll Brothers Inc	Operative Builders	3.33%	Baa3
TSG	Sabre Holdings Corp	CMP Integrated Sys Design	3.33%	B1
TWX	Time Warner Inc	Cable And Other Pay TV Svcs	3.33%	Baa1
WHR	Whirlpool Corp	Household Appliances	3.33%	Baa2

Table 2. Parameters Estimations for State Factor Process

This table shows the results of parameter estimations in equation (2) by MLE method. The input data are the historical state factor values obtained by the factor analysis method on firms' free cash flows. We fit the data to equation (2). The equation is adopted because it is empirically proven that macroeconomic factors are weakly stationary processes. We utilize maximum likelihood algorithm to calibrate the parameters of the mean-reverting speed (a), long-term average level (b), and the standard deviation of state's term variation (σ).

Parameters	Mean-reverting speed a_F	Long-term average b_F	S.D. of variation σ_F
Factor 1	0.1000	0.0003	0.3688
Factor 2	0.1039	0.3134	0.4451
Factor 3	0.1134	-0.2860	0.4643
Factor 4	0.2290	0.0213	0.6310
Factor 5	0.2676	-0.0365	0.6929
Factor 6	0.5368	-0.0386	0.9870
Factor 7	0.6232	-0.0529	1.0910

Table 3. The Estimation Results of Each Firm's WACC

This table shows each component firm's estimated WACC by the weighted average of the cost of capital and the cost of debt. The weight of the former and the latter is one minus debt ratio and debt ratio, respectively. The detailed needed data descriptions are shown in the part D.2.

Ticker	Individual firm's beta	Moody's Credit Rating	WACC(Y)	WACC(Q)
ARW	1.9303	Baa3	12.52%	3.13%
AZO	0.7150	Baa1	6.08%	1.52%
CBS	0.9386	Baa2	9.12%	2.28%
CCU	1.4272	B1	15.30%	3.83%
CTL	1.1988	Baa2	9.32%	2.33%
EQ	0.2969	Baa3	7.09%	1.77%
EXPE	2.0592	Ba1	16.85%	4.21%
GPS	1.3780	Ba1	12.50%	3.12%
HET	0.5780	Ba2	10.16%	2.54%
HOT	1.0635	Baa3	8.90%	2.22%
IACI	1.8366	Baa3	14.66%	3.66%
IP	0.8403	Baa2	7.68%	1.92%
JNY	0.9708	Baa3	9.95%	2.49%
LEN	0.6903	Baa2	7.80%	1.95%
LTD	0.9259	Baa2	8.37%	2.09%
MWV	1.1018	Baa2	8.67%	2.17%
OLN	0.8218	Baa3	8.34%	2.09%
PHM	0.9902	Baa2	9.08%	2.27%
RRD	0.4174	Baa1	6.57%	1.64%
RSH	1.1160	Ba2	11.37%	2.84%
SHW	0.9265	A3	7.84%	1.96%
SWY	0.8976	Baa3	8.60%	2.15%
TIN	1.5598	Baa2	7.14%	1.78%
TOL	1.3909	Baa3	10.71%	2.68%
TSG	1.8747	B1	16.82%	4.20%
TWX	1.9282	Baa1	11.83%	2.96%
WHR	1.1785	Baa2	7.81%	1.95%

Table 4. The Estimation Results of Each Firm's Implied Cash Flow Shift Term

This table shows each component firm's implied cash flow shift term per asset book value that can reconcile the present value of future cash flows with the current market asset value. According to (20), we can calibrate the implied cash flow shift term (m) by optimization techniques when the values of other variables and parameters are determined. The detailed needed data descriptions are shown in the part D.

Ticker	Asset value ($mm)	WACC(Q)	Implied Cash Flow Shift Term
ARW	7409.15	3.13%	0.0230
AZO	12191.03	1.52%	-0.0176
CBS	21950.76	2.28%	0.0000
CCU	27836.90	3.83%	0.0396
CTL	8992.12	2.33%	0.0000
EQ	16802.61	1.77%	0.0096
EXPE	8426.00	4.21%	0.0384
GPS	19366.10	3.12%	0.0337
HET	30481.62	2.54%	0.0215
HOT	19486.00	2.22%	0.0271
IACI	14082.97	3.66%	0.0258
IP	30771.00	1.92%	0.0091
JNY	5198.05	2.49%	0.0008
LEN	7928.61	1.95%	-0.0050
LTD	15902.30	2.09%	0.0212
MWV	10896.69	2.17%	0.0132
OLN	5169.95	2.09%	0.0417
PHM	14746.50	2.27%	0.0269
RRD	12979.80	1.64%	-0.0040
RSH	3665.54	2.84%	0.0151
SHW	11356.15	1.96%	-0.0013
SWY	24917.90	2.15%	0.0107
TIN	22049.06	1.78%	0.0028
TOL	8878.27	2.68%	0.0274
TSG	6303.85	4.20%	0.0479
TWX	152898.09	2.96%	0.0141
WHR	16676.45	1.95%	0.0026

Table 5. Parameter Estimation of the Portfolio Loss Distribution Fittings

We fit the portfolio loss distribution to the specified distributions mentioned in literature. The value in parenthesis is the standard deviation of the parameter. The results of the parameter estimates and their standard deviations show that the distributions fittings are efficient. Panel A and B separately show that the parameters estimates of the fitted distributions with constant and dynamic default threshold settings.

Distribution Fitted	Gamma		Birnbaum-Saunders		Weibull		Beta	
Parameters	a: shape	b: scale	beta: scale	gamma: shape	a: scale	b: shape	a	b
Panel A. The Parameters Estimates of the Fitted Distributions (Constant Default Threshold)								
Estimates	2.056	0.002	0.003	0.654	0.005	1.235	2.038	489.902
(Std. Err.)	(0.086)	(0.000)	(0.000)	(0.015)	(0.000)	(0.026)	(0.155)	(20.719)
Panel B. The Parameters Estimates of the Fitted Distributions (Dynamic Default Threshold)								
Estimates	0.695	0.020	0.007	1.457	0.011	0.757	0.685	48.736
(Std. Err.)	(0.003)	(0.000)	(0.000)	(0.003)	(0.000)	(0.002)	(0.005)	(0.335)

Table 6. The Empirical Examinations for the 5-year NA.IG.HVOL.CDX

Table 6 contains two panels. Panel A shows the pricing results of the cash flow-based credit portfolio models with the constant and dynamic default thresholds. Panel B summarizes the calculated spreads in panel A and compares them with the market spreads during the end month and the all months in 2006. The results show that the model with dynamic default threshold performs so well.

Panel A. The Empirical Pricing Results of the Cash Flow-based Model		
Scenario / Items	Model Pricing (Constant Default Threshold)	Model Pricing (Dynamic Default Threshold)
With Market-traded CDS	36.39 bps	84.25 bps
With Moody's idealized expected loss rate	41.37 bps	89.23 bps
Panel B. The Comparisons between the Model Spreads and Market Spreads		
Range of Model Spreads	(36.39, 41.37) bps	(84.25, 89.23) bps
Market Spreads (Dec)	(79.75, 84.25) bps	(79.75, 84.25) bps
Market Spreads (All Year)	(64.00, 98.50) bps	(64.00, 98.50) bps

Appendix I. Derivations of the Marginal Expected Recovery Rate Equations

Similarly, equation (16) and (17) in C.2 gives the equation of the marginal expected recovery rate in the model with the stationary default threshold. In the following, we show the details of its derivations.

If $0 < \ell V_{t-1} < \hat{L}_t$,

$$PD_t E\left(\frac{V_t}{L_t} \middle| L_t > V_t\right) = \begin{cases} \Gamma_t^{'}, & if \;\; 0 < V_t < \ell V_{t-1} \\ \Gamma_t^{''}, & if \;\; \ell V_{t-1} < V_t < V_{t-1} \end{cases} \tag{A.1}$$

where

$$\begin{aligned} \Gamma_t^{'} &= \int_{V_t}^{\hat{L}_t} \frac{V_t}{x} f(x) dx \\ &= \int_{V_t}^{\ell V_{t-1}} \frac{V_t}{x} \frac{2}{\ell V_{t-1} \hat{L}_t} x dx + \int_{\ell V_{t-1}}^{\hat{L}_t} \frac{V_t}{x} \frac{2(\hat{L}_t - x)}{\hat{L}_t(\hat{L}_t - \ell V_{t-1})} dx \\ &= \frac{2V_t(\ell V_{t-1} - V_t)}{\hat{L}_t \ell V_{t-1}} + \frac{2V_t}{\hat{L}_t(\hat{L}_t - \ell V_{t-1})}\left(\hat{L}_t \ln \frac{\hat{L}_t}{\ell V_{t-1}} - \hat{L}_t + \ell V_{t-1}\right) \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} \Gamma_t^{''} &= \int_{V_t}^{\hat{L}_t} \frac{V_t}{x} f(x) dx \\ &= \int_{V_t}^{\hat{L}_t} \frac{V_t}{x} \frac{2(\hat{L}_t - x)}{\hat{L}_t(\hat{L}_t - \ell V_{t-1})} dx \\ &= \frac{2V_t}{\hat{L}_t(\hat{L}_t - \ell V_{t-1})}\left(\hat{L}_t \ln \frac{\hat{L}_t}{V_t} - \hat{L}_t + V_t\right) \end{aligned} \tag{A.3}$$

Plugging in the results above, we can derive the equation (16) and (17).

If $V_t < \hat{L}_t$ and $0 < \ell V_{t-1} < \hat{L}_t$,

$$\Lambda_t = (1 - PD_t) + PD_t E\left(\frac{V_t}{L_t} \mid L_t > V_t\right) \tag{16}$$

$$= \begin{cases} \Lambda_t' = \dfrac{V_t^2 + 2V_t(\ell V_{t-1} - V_t)}{\hat{L}_t \cdot \ell V_{t-1}} + \dfrac{2V_t}{\hat{L}_t(\hat{L}_t - \ell V_{t-1})}\left(\hat{L}_t \ln \dfrac{\hat{L}_t}{\ell V_{t-1}} - \hat{L}_t + \ell V_{t-1}\right), & if \; 0 < V_t < \ell V_{t-1} \\ \Lambda_t^{''} = 1 - \dfrac{(\hat{L}_t - V_t)^2}{\hat{L}_t(\hat{L}_t - \ell V_{t-1})} + \dfrac{2V_t}{\hat{L}_t(\hat{L}_t - \ell V_{t-1})}\left(\hat{L}_t \ln \dfrac{\hat{L}_t}{V_t} - \hat{L}_t + V_t\right), & if \; \ell V_{t-1} < V_t < \hat{L}_t \end{cases}$$

If $V_t < \hat{L}_t$ and $\ell V_{t-1} > \hat{L}_t$,

$$\begin{aligned}\Lambda_t &= (1-PD_t) + PD_t E(\frac{V_t}{L_t} \mid L_t > V_t) \\ &= \left(\frac{V_t}{\hat{L}_t}\right)^2 + \frac{2V_t\left(\hat{L}_t - V_t\right)}{\hat{L}_t^{\,2}}\end{aligned} \tag{17}$$