VaR Estimation under Stochastic Volatility Models

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Outline

- Risk Management in Practice: Value at Risk (VaR)

- Estimate Default Probability by Efficient Importance Sampling

- Fourier Transform Method: boundary effect and a price correction scheme

- Stability of Estimation and some Empirical Results

Value at Risk

Let $r(t)$ be an asset return at time t. Its $(1-\alpha)\%$ VaR, denoted by VaR_α, is defined by the $\alpha\%$-quantile of $r(t)$. That is,

$$\mathbf{P}\left(r(t) \geq -\mathbf{VaR}_\alpha\right) = 1 - \alpha.$$

That is a risk controller has a $(1-\alpha)\%$ confidence that the asset price will not drop below VaR_α in time t.

Aspects about VaR

- Mathematically, it is not a **coherent** risk measure* because it doesn't satisfy the risk diversification principal. Instead, CVaR is.

- Practically, it is commonly required by financial regulations.

*Artzner P., F. Delbaen, J.-M. Eber, and D. Heath, "Coherent Measures of Risk," *Mathematical Finance,* 9 (1999): 203-28.

Estimation of VaR

- Riskmetrics: normal assumption
- Historical Simulation: generate scenarios
- Monte Carlo method: model dependent

Estimate Probability of Default

Given a dynamical model of an asset price S_t, its return process is $r_t = \ln S_t/S_{t-1}$.

Given a loss threshold B, the probability of default is defined by

$$\text{DP(B)} = E_{t-1}\left\{\mathbf{I}(r(t) \le -B)\right\}.$$

Note: VaR_α is the B satisfying $DP(B) = \alpha$.

Importance Sampling

Given the Black Scholes Model under measure P, a new measure $\tilde{P}$ defined from an exponential martingale $\frac{dp}{d\tilde{P}} = Q$ satisfies

$$\tilde{E}_{t-1}\left[S_t\right] = \exp(-B).$$

Denote DP by P_ε and the second moment by $M_{2\varepsilon}$, which are defined by

$$\begin{aligned} P_\varepsilon &= \mathbb{E}_{t-1}\left[\mathbf{I}\left(r_t \leq -B\right)\right] \\ M_{2\varepsilon} &= \tilde{\mathbb{E}}_{t-1}\left[\mathbf{I}\left(r_t \leq -B\right) Q^2\right]. \end{aligned}$$

Asymptotic Optimality in Variance Reduction

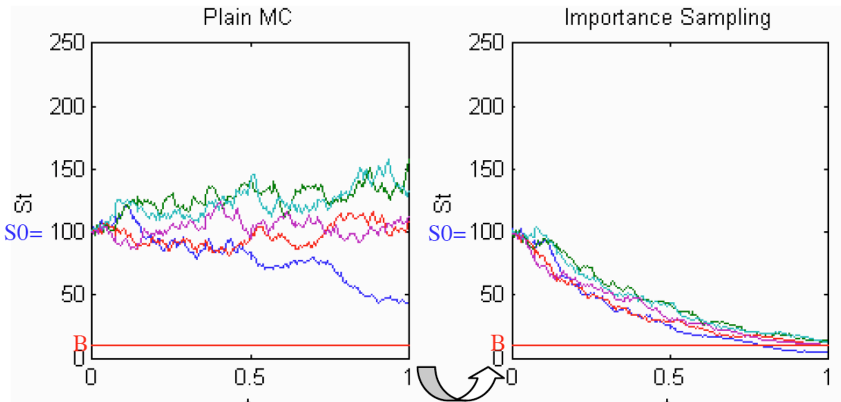
Theorem: $M_{2\varepsilon} \approx (P_\varepsilon)^2$ for **small** ε (spatial scale). Thus, the importance sampling is optimal (or efficient).

Proof: by means of Cramer's theorem for 1-dim case.

For high-dimensional first passage time problem, see H. (09).

Trajectories under different measures
Single Name Case

Simulation of the stock price :



$$\frac{dP^*}{d\widetilde{P}} = Q_T$$

Some Modifications: SV model and Jump-Diffusion Model

SV Model:

$$\begin{cases} dS_t = \mu\, S_t\, dt + \sigma_t\, S_t\, dW_t \\ \sigma_t = \exp(Y_t/2) \\ dY_t = \ (m - Y_t)\, dt + \beta\, dZ_t \end{cases}$$

JD Model:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + d\left(\sum_{j=1}^{N_t} (Y(j) - 1)\right),$$

1-dim. Default Probability - SV Model

B	BMC	Importance Sampling
94.855	0.0103 (0.0010)	0.0099 (1.6964E-004)
96.36	0.0501 (0.0022)	0.0500 (7.3140E-004)

The number of simulations is 10^4 and the Euler discretization takes time step size $T/100$, where T is one day.

Other parameters are $S_0 = 100, \mu = 0.3, m = -2, \alpha = 5, \beta = 1, \rho = 0$. Standard errors are shown in parenthesis.

Default Probability - Jump-Diffusion Model

B	P_{JD} True	P_{JD} Basic MC	P_{JD} IS-JD*	P_{JD} IS-D
0.0211	0.05	0.0499 (0.0069)	0.0501 (0.0024)	0.0482 (7.89×10^-4)
0.0298	0.01	0.01 (3.1×10^-3)	0.01 (6.8×10^-4)	0.0094 (1.7084×10^-4)
0.04	0.001	0.0001 (9.9×10^-4)	0.001 (1×10^-4)	0.00095 (8.38×10^-5)

The number of simulations is 10^4 and the Euler discretization takes time step size $T/100$, where T is one day.

Other parameters are $\mu = 0.06, \sigma = 0.2, \lambda = 1, a = 0, b^2 = 0.02, T = 1/252$. Standard errors are shown in parenthesis.

A Nonparametric Method to Estimate Vol.
Fourier Transform Method*

Assume a difussion process

$$du(t) = \mu(t)dt + \sigma(t)dW_t,$$

Task: to estimate is $\sigma(t)$, i.e. the time series volatility.

*Malliavin and Mancino(2002,2005,2009)

Fourier Transform Method(Step 1)

Compute the Fourier coefficients of du by

$$
\begin{aligned}
a_0(du) &= \frac{1}{2\pi}\int_0^{2\pi} du(t), \\
a_k(du) &= \frac{1}{\pi}\int_0^{2\pi} \cos(kt)du(t), \\
b_k(du) &= \frac{1}{\pi}\int_0^{2\pi} \sin(kt)du(t).
\end{aligned}
$$

Then,

$$
u(t) = a_0 + \sum_{k=1}^{\infty}\left[-\frac{b_k(du)}{k}\cos(kt) + \frac{a_k(du)}{k}\sin(kt)\right].
$$

Fourier Transform Method(Step 2)

Fourier coefficients of variance σ^2,

$$a_0(\sigma^2) = \lim_{N\to\infty} \frac{\pi}{N+1-n_0} \sum_{s=n_0}^{N} \left[a_s^2(du) + b_s^2(du)\right],$$

$$a_k(\sigma^2) = \lim_{N\to\infty} \frac{2\pi}{N+1-n_0} \sum_{s=n_0}^{N} \left[a_s(du)a_{s+k}(du)\right], \forall k > 0,$$

$$b_k(\sigma^2) = \lim_{N\to\infty} \frac{2\pi}{N+1-n_0} \sum_{s=n_0}^{N} \left[a_s(du)b_{s+k}(du)\right], \forall k \geq 0,$$

where n_0 is any positive integer. so that

$$\sigma_N^2(t) = \sum_{k=0}^{N} \left[a_k(\sigma^2)\cos(kt) + b_k(\sigma^2)\sin(kt)\right].$$

Fourier Transform Method(Step 3)

Reconstruct the time series variance $\sigma^2(t)$.

- Finally, $\sigma_N^2(t)$ is an approximation of $\sigma^2(t)$ as N approaches infinity, which can be given by classical Fourier-Fejer inversion formula.

$$\sigma^2(t) = \lim_{N\to\infty} \sigma_N^2(t) \text{ in prob.}$$

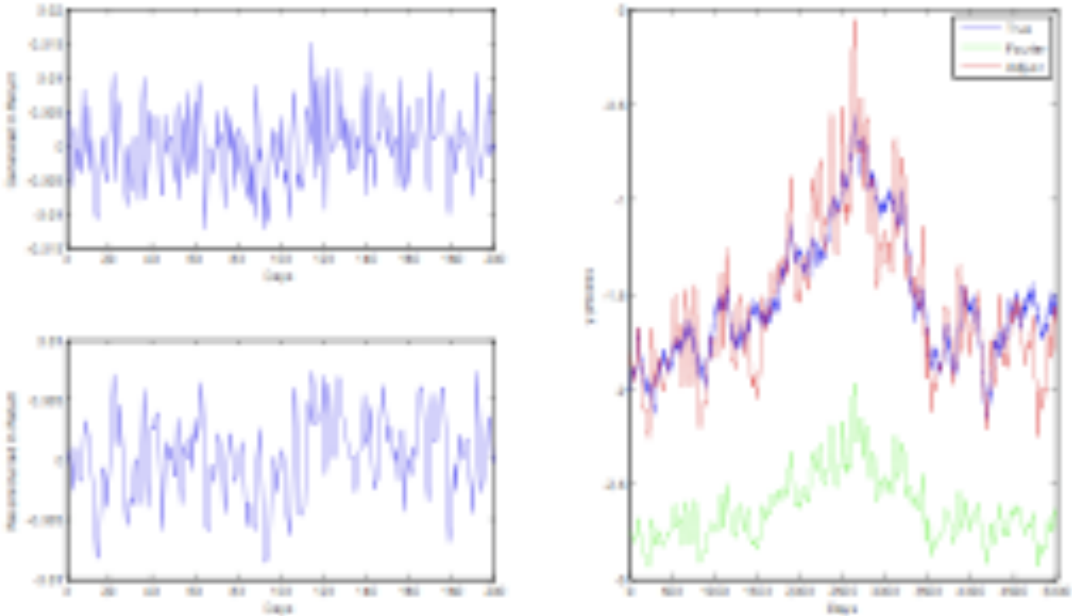
Smoothing

- We add a function into the final computation of time series variance in order to smooth it.

$$\sigma^2(t)$$
$$= \lim_{N\to\infty} \sum_{k=0}^{N} \varphi(\delta k) \left[a_k(\sigma^2)\cos(kt) + b_k(\sigma^2)\sin(kt) \right]$$

where $\varphi(x) = \frac{\sin^2(x)}{x^2}$ is a function in order to smooth the trajectory and δ is a smoothing parameter.

Boundary Effect Removed
Simulated Data



A Price Correction Scheme: First Order

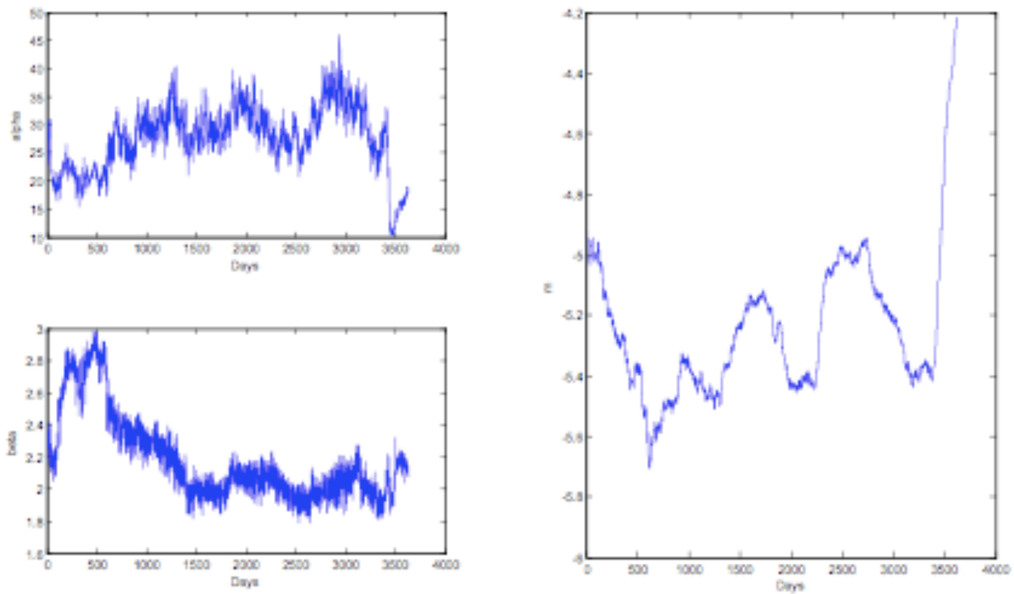
Idea: (Nonlinear) Least Squares Method for first-order correction

$$
\begin{aligned}
r_t &\approx \sigma_t \, \delta_t \, \epsilon_t \\
&\approx \exp\left(\left(a + b\,\widehat{Y}_t\right)/2\right) \delta_t \, \epsilon_t.
\end{aligned}
$$

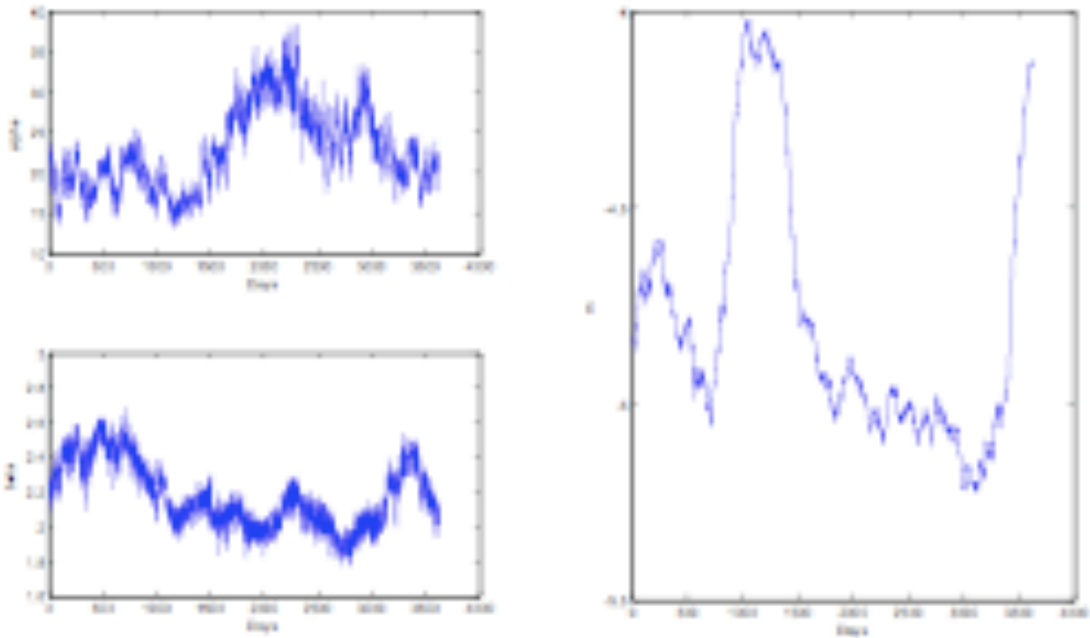
Then by MLE to regress out a and b

$$
\ln \frac{r_t^2}{\delta_t^2} = a + b\,\widehat{Y}_t + \ln \epsilon_t^2.
$$

Stability of parameters - GBP/USD



Stability of parameters - JPY/USD



Back Testing
Empirical LRcc Test

Data Sample Period: 1993/1/5 $\sim$ 2009/7/24

1% VaR	RiskMetrics	H.S.	SV Model
AUD	O	X	O
JPY	X	X	O
SGD	X	X	X
CAD	X	X	O
KRW	X	X	X
GBP	X	X	O

Conclusion

- Simple and efficient importance sampling methods are proposed, justified by large deviation theory.
- Remove boundary effect of Fourier Transform Method
- Some empirical studies on FX data

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Thank You